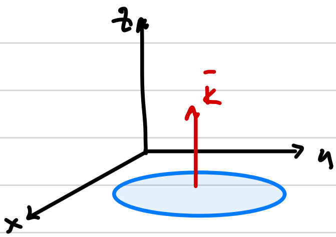
Vector Forms of Green's Thm

Recall : Green's Thm $\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

Vector Form #1 : (Consider Stokes' thm)



Embed C, D into $\mathbb{R}^3$ and consider

$$\bar{F} = (M(x,y),\, N(x,y),\, 0)$$

Then $\operatorname{curl} \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & 0 \end{vmatrix}$

$$= \left(0,\ 0,\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

N and M depend only on x, y.

and $\operatorname{curl} \bar{F} \cdot \bar{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$

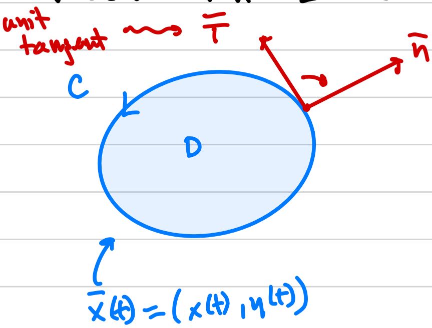
So : Green's can be reinterpreted :

$$\oint_C \bar{F} \cdot d\bar{s} = \oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_D \operatorname{curl} \bar{F} \cdot \bar{k}\, dA$$

notation change Green's above

↳ $\oint_C \bar{F} \cdot d\bar{s}$ measures tendency of $\bar{F}$ to "swirl" (or curl!) around axis defined by $\bar{k}$.

Vector Form #2: (Consider Divergence Thm)



Since $\bar{T} = \dfrac{\bar{x}'(t)}{\|\bar{x}'(t)\|}$

$$= \left(\frac{x'(t)}{\|\bar{x}'(t)\|}, \frac{y'(t)}{\|\bar{x}'(t)\|} \right)$$

can rotate 90° clockwise to get

orthogonal... check dot product 0

$$\bar{n}(t) = \left(\frac{y'(t)}{\|\bar{x}'(t)\|}, \frac{-x'(t)}{\|\bar{x}'(t)\|} \right)$$

Note: $\oint_C \bar{F} \cdot \bar{T} \, ds$ measures tendency of $\bar{F}$ to push along C.

flux of F

$\oint_C \bar{F} \cdot \bar{n} \, ds$ measures tendency of $\bar{F}$ to push across C.

arc length

So: $\oint_C \underbrace{\bar{F} \cdot \bar{n}}_{\text{scalar function}} \, ds = \int_a^b \bar{F} \cdot \bar{n} \, \|\bar{x}'(t)\| \, dt$ (how to compute $\int_C f \, ds$)

$$= \int_a^b (M, N) \cdot \left(\frac{y'(t)}{\|\bar{x}'(t)\|}, \frac{-x'(t)}{\|\bar{x}'(t)\|} \right) \|\bar{x}'(t)\| \, dt \quad \text{(substitute)}$$

$$= \int_a^b M y'(t) - N x'(t) \, dt \quad \text{(cancel and dot product)}$$

$$= \oint_C -N \, dx + M \, dy \quad \text{(interpretation of } \textstyle\int_C f \, dx \text{ or } \int_C f \, dy)$$

Green's thm!!

$$= \iint_D \frac{\partial M}{\partial x} - \left(-\frac{\partial N}{\partial y}\right) dA$$

$$= \iint_D \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\, dA$$

$\bar{F} = (M, N)$

$\operatorname{div} \bar{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$

$= \nabla \cdot \bar{F}$

$$= \iint_D \operatorname{div} \bar{F}\, dA$$

So: flux $\left(\oint_C \bar{F} \cdot \bar{n}\, ds\right)$ is integral of divergence.

This is why we interpret $\operatorname{div} \bar{F}$ as tendency to push out/away from point.