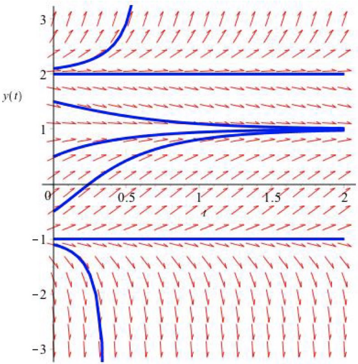
MATH 226: Differential Equations



February 14, 2025


Happy
Valentine's
Day

Notes on Assignment 1
Assignment 2
Euler Method Approximations in *Maple* and MATLAB

Announcements

1. Homework Grader:
   Halima Dieye
2. Explore Course Website
3. Practice Problems vs. Feedback Problems

Key Terms From Chapter 1

Independent Variable
Dependent Variable
Parameter
Solution
Equilibrium Solution
Integral Curves
Autonomous Differential Equation
Critical Point = Fixed Point = Stationary Point
Phase Line
One - Dimensional Phase Portrait
Asymptotically Stable
Unstable
Semistable Attractor = Sink
Repeller = Source
Linearization About An Equilibrium
Direction Field

VERIFYING A PROPOSED SOLUTION

Example: Verify that $y = Ce^{-4t} + \frac{1}{4}t + \frac{1}{2}e^{-2t} - \frac{1}{16}$ is a solution of

$$y' + 4y = t + e^{-2t}$$

Verification

$$y' = -4Ce^{-4t} + \frac{1}{4} - e^{-2t} - 0$$

$$4y = 4Ce^{-4t} + t + 2e^{-2t} - \frac{1}{4}$$

Adding:

$$y' + 4y = t + e^{-2t}$$

The Differential Equation

$$P'(t) = kP(t)$$

Has Solution (if k is a constant)

$$P(t) = P(0)e^{kt}$$

We can write $P'(t) = kP(t)$ as $P' = kP$

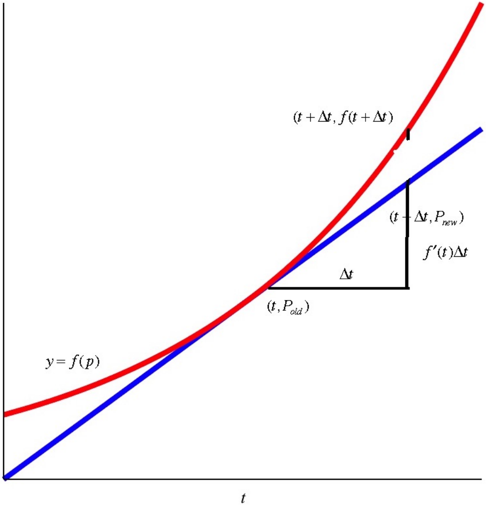
We change the name of the dependent variable:

$$y' = ky \Rightarrow y = y(0)e^{kt}$$

$$x' = kx \Rightarrow x = x(0)e^{kt}$$

Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$



Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$

Example: $k = .04, P(0) = 1000$

Time	Approximate	Exact
0.	1000.00	1000.00
0.1000000000	1004.00000	1004.008011
0.2000000000	1008.016000	1008.032086
0.3000000000	1012.048064	1012.072289
0.4000000000	1016.096256	1016.128685
0.5000000000	1020.160641	1020.201340
0.6000000000	1024.241284	1024.290318
0.7000000000	1028.338249	1028.395684
0.8000000000	1032.451602	1032.517505
0.9000000000	1036.581408	1036.655846
1.000000000	1040.727734	1040.810774

A MATLAB Program for Euler Method

```
clc
clear
dt=0.1;
t=0:dt:1.0;
p(1)=1000; % p(0)=1
for i=1:length(t) -1
   p(i+1)=p(i)+dt*.04*p(i);
   fprintf('Approximate value at  t=%f is %f; exact value is %f\n', i/10, p(i+1), 1000* exp(.04 *
i/10)  );
end
```

Here is output:

```
Approximate value at  t=0.100000 is 1004.000000; exact value is 1004.008011
Approximate value at  t=0.200000 is 1008.016000; exact value is 1008.032086
Approximate value at  t=0.300000 is 1012.048064; exact value is 1012.072289
Approximate value at  t=0.400000 is 1016.096256; exact value is 1016.128685
Approximate value at  t=0.500000 is 1020.160641; exact value is 1020.201340
Approximate value at  t=0.600000 is 1024.241284; exact value is 1024.290318
Approximate value at  t=0.700000 is 1028.338249; exact value is 1028.395684
Approximate value at  t=0.800000 is 1032.451602; exact value is 1032.517505
Approximate value at  t=0.900000 is 1036.581408; exact value is 1036.655846
Approximate value at  t=1.000000 is 1040.727734; exact value is 1040.810774
```

Exact Solution and Euler Approximation in *Maple*

$ode := P'(t) = .04 \cdot P(t)$

$$ode := \mathrm{D}(P)(t) = 0.04\, P(t) \qquad (1)$$

$init := P(0) = 1000$

$$init := P(0) = 1000 \qquad (2)$$

$dsolve(\ \{ode, init\}\ )$

$$P(t) = 1000\, \mathrm{e}^{\frac{t}{25}} \qquad (3)$$

$A[0] := 1000.0$

$$A_0 := 1000.0 \qquad (4)$$

for i **from** 1 **to** 10 **do** $A[i] := A[i-1] + .04 \cdot A[i-1] \cdot 0.1$ **od**:

for t **from** 0 **to** 10 **do**
  if $t = 0$ **then** *print*('*Time*', '*Approximation*', '*Exact*') **fi** ;
  $print\left(\frac{t}{10.}, A[t], 1000 \cdot \exp\left(\frac{t}{250.0}\right)\right)$
od;

Time, *Approximation*, *Exact*

0., 1000.0, 1000.

0.1000000000, 1004.0000, 1004.008011

0.2000000000, 1008.016000, 1008.032086

0.3000000000, 1012.048064, 1012.072289

0.4000000000, 1016.096256, 1016.128685

0.5000000000, 1020.160641, 1020.201340

0.6000000000, 1024.241284, 1024.290318

0.7000000000, 1028.338249, 1028.395684

0.8000000000, 1032.451602, 1032.517505

0.9000000000, 1036.581408, 1036.655846

1.000000000, 1040.727734, 1040.810774 (5)

Generalizations

1. Population with immigration and/or emigration
2. Forest Management
3. Fishery Management
4. Lake Champlain Pollution
5. Anesthetic
6. Alcohol/Drug

$$P' = aP + b$$

More Complicated Relationships

1. Population with immigration and/or emigration
2. Forest Management
3. Fishery Management
4. Lake Champlain Pollution
5. Anesthetic
6. Alcohol/Drug
7. Credit Card Debt

$$P' = aP + b$$

where a and b are constants

$$P' = aP + b \text{ or } y' = ay + b$$

where a and b are constants

Suppose 5 percent breaks down every month and 1000 units per month flow in.
Let y be amount (in tons) in lake at time t months.

$$y'(t) = -\frac{1}{20}y(t) + 1000 \text{ or } y' = -\frac{1}{20}y + 1000$$

Qualitative Analysis

Begin by looking for Constant Solutions.
What is true about constant functions?
The derivative is always 0.
Find where $y' = 0$.

$$-\frac{1}{20}y + 1000 = 0$$

$$\frac{1}{20}y = 1000$$

$$y = (20)(1000) = 20,000$$

$$y' = 0 \text{ when } y = 20,000$$

Equilibrium, Critical Point, Stationary Point

$$y'(t) = -\frac{1}{20}y(t) + 1000 \text{ or } y' = -\frac{1}{20}y + 1000$$

Qualitative Analysis II

When is y increasing and when is y decreasing?

y increases when y' is positive.

When is $y' > 0$?

Qualitative Analysis of

$$y'(t) = -\frac{1}{20}y(t) + 1000$$

$$\underline{y' > 0}$$

$$-\frac{1}{20}y + 1000 > 0$$

$$1000 > \frac{1}{20}y$$

$$20,000 > y$$

y is increasing when y is less than 20,000.
Similarly: y is decreasing when y is greater than 20,000.
$y' = 0$ when $y = 20,000$ [**Equilibrium, Critical Point, Stationary Point**]

Analytic Solution of $y'(t) = -\frac{1}{20}y(t) + 1000$

Method I: Divide each side by $-\frac{1}{20}y + 1000$ and integrate:

$$\int \frac{1}{-\frac{1}{20}y + 1000}dy = \int 1dt$$

Method II: Write equation as

$$y' = -\frac{1}{20}[y + \frac{1000}{-\frac{1}{20}}] = -\frac{1}{20}(y - 20,000)$$

and make **Change of Variable** $P = y - 20,000$:
Then $P' = y'$ so the equation becomes

$$P' = -\frac{1}{20}P$$

which has solution $P(t) = P(0)e^{-\frac{1}{20}t}$

$$P' = -\frac{1}{20}P \text{ has solution } P(t) = P(0)e^{-\frac{1}{20}t}$$

where

$$P = y - 20000 \text{ so } y = P + 20000$$

Thus the solution to

$$y'(t) = -\frac{1}{20}y(t) + 1000$$

$$\text{is } y - 20,000 = [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{or } y = 20,000 + [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{Since } \lim_{t\to\infty} e^{-\frac{1}{20}t} = 0$$

$$y \to 20,000 \text{ as } t \to \infty$$

Example: If $y(0) = 30,000$, find T so $y(T) = 20,001$.

$$y(t) = 20,000 + [y(0) - 20,000]e^{-\frac{1}{20}t}$$

$$\text{so } 20,001 = 20,000 + [30,000 - 20,000]e^{-\frac{1}{20}T}$$

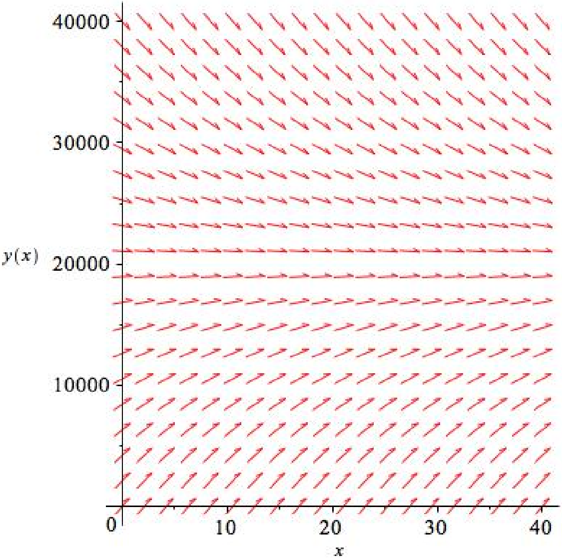
$$1 = 10,000e^{-\frac{1}{20}T}$$

$$e^{-\frac{1}{20}T} = \frac{1}{10,000}$$

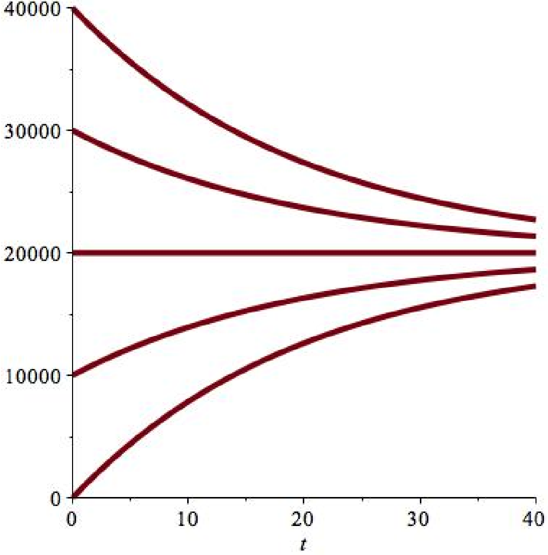
$$-\frac{1}{20}T = \ln(\frac{1}{10,000}) = -\ln 10,000$$

$$T = 20\ln 10,000 = 20\ln 10^4 = 20(4)\ln 10 \sim 184.2$$

Direction Field for $P' = -.05P + 1000$



Some Integral Curves For $P' = -.05P + 1000$



$$\text{Observe } y' = -\frac{1}{20}y + 1000$$

has form $y' = ay + b$ where a, b are constants.
We can solve in the same way:

$$y' = a(y + \frac{b}{a})$$

$$\text{Let } P = y + \frac{a}{b} \text{ so } P' = y'$$

$$\text{We have } P' = aP \text{ so } P = P(0)e^{at}$$

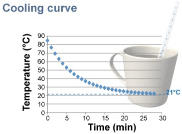
$$\text{or } y + \frac{b}{a} = [y(0) + \frac{b}{a}]e^{at}$$

$$y = [y(0) + \frac{b}{a}]e^{at} - \frac{b}{a}$$

WE NOW KNOW HOW TO SOLVE

$$y' = ay + b$$

where a and b are constants,
This equation has been used to model many different situations.



Newton's Law of Cooling (T is constant ambient temperature)

$$y' = -k(y - T) = -ky + kT$$

Mice and Owls:

$$P' = 0.5P - 450$$

A Major Generalization

$$y' = ay + b$$

There is no explicit t on right hand side.

More generally, a Differential Equation of the form

$y' = f(y)$ where f is a function only of y

is called an **Autonomous Equation**