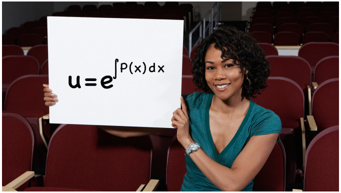
MATH 226: Differential Equations



February 21, 2025

Notes on Assignment 3
Assignment 4

First Order LINEAR Differential Equations

$$\frac{dy}{dt} + p(t)y = g(t)$$

Method of Integrating Factors

First Order LINEAR Differential Equations

$$\frac{dy}{dt} + p(t)y = g(t)$$

$$y' + p(t)y = g(t)$$

- y' and y appear all by themselves
- No terms like y^2 or $\cos y$ or $y\ y'$
- $p(t)$ and $g(t)$ can be nonlinear, complicated, but continuous.

First Order LINEAR Differential Equations

$$y' + p(t)y = g(t)$$

Solution By Method of Integrating Factors:

Multiply Equation By Factor That Converts Left Hand Side Into a Derivative

Fundamental Theorem of Calculus

$$\text{If } f(t) = \int p(t)dt, \text{ then } f'(t) = p(t)$$

$$\text{If } f(t) = \int_0^t p(s)ds, \text{ then } f'(t) = p(t)$$

Application:

Find the derivative of $e^{\int p(t)\,dt} = exp(\int p(t)\,dt)$ with respect to t

Solution : Use Product Rule

$$\left(e^{\int p(t)\,dt}\right)' = e^{\int p(t)\,dt} p(t) = p(t) e^{\int p(t)\,dt}$$

Let's Do a Specific Example (Problem 15 in Text):

$$\text{Solve } ty' + 4y = t^2 - t + 1, \text{ with } y(1) = \frac{1}{4}$$

$$\text{Put in Standard Form (*): } y' + \frac{4}{t}y = t - 1 + \frac{1}{t}$$

$$\text{Integrating Factor is } e^{\int \frac{4}{t}\,dt} = e^{4\ln t} = e^{\ln t^4} = t^4$$

$$\text{Multiply (*) by } t^4 : t^4y' + 4t^3y = t^5 - t^4 + t^3$$

$$(t^4y)' = t^5 - t^4 + t^3 \text{ and integrate}$$

$$t^4y = \frac{t^6}{6} - \frac{t^5}{5} + \frac{t^4}{4} + C$$

$$\text{Solve for } y : y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + Ct^{-4}$$

Original Initial Value Problem

Solve $ty' + 4y = t^2 - t + 1,$ with $y(1) = \frac{1}{4}$

Solution

$$y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{C}{t^4}$$

Use initial condition to find C:

$$\frac{1}{4} = \frac{1}{6} - \frac{1}{5} + \frac{1}{4} + C$$

$$C = \frac{1}{30}$$

so

$$y = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{1}{30t^4}$$

General Case:

$$y' + p(t)y = g(t)$$

Multiply through by $e^{\int p(t)\,dt}$:

$$e^{\int p(t)\,dt}y' + p(t)e^{\int p(t)\,dt}y = g(t)e^{\int p(t)\,dt}$$

Rewrite Left Hand Side:

$$\left(e^{\int p(t)\,dt}y\right)' = g(t)e^{\int p(t)\,dt}$$

Integrate Both Sides:

$$e^{\int p(t)\,dt}y = \int g(t)e^{\int p(t)\,dt} + C$$

Divide by Coefficient of y:

$$y = e^{-\int p(t)\,dt}\int g(t)e^{\int p(t)\,dt} + Ce^{-\int p(t)\,dt}$$

$$y' + p(t)y = g(t) \text{ has solution}$$

$$y = e^{-\int p(t)\,dt} \int g(t) e^{\int p(t)\,dt} + C e^{-\int p(t)\,dt}$$

Problem 35: Construct a first order linear differential equation whose solutions are asymptotic to the line $y = 4 - t$ as $t \to \infty$.

Solution: Add a term that goes to 0 as $t \to \infty$.

One choice would be Ce^{-t} for an arbitrary constant C.

Then solution has form $y = Ce^{-t} + 4 - t$.

Differentiate with respect to t: $y' = -Ce^{-t} - 1$

But $Ce^{-t} = y - 4 + t$

So $y' = -(y - 4 + t) - 1 = -y + 4 - t - 1 = -y + 3 - t$

$$y' + y = 3 - t$$

Next Time

Modeling With First Order Differential Equations