MATH 226: Differential Equations

February 26, 2025

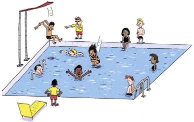
- Team Assignments: Project One
- Assignment 6

Announcements

First Team Project

Due: Week From Friday

Meet With Me



7. An outdoor swimming pool loses 0.05% of its water volume every day it is in use, due to losses from evaporation and from excited swimmers who splash water. A system is available to continually replace water at a rate of G gallons per day of use.

(a) Find an expression, in terms of G, for the equilibrium volume of the pool. Sketch a few graphs for the volume $V(t)$, including all possible types of solutions.
(b) If the pool volume is initially 1% above its equilibrium value, find an expression for $V(t)$.
(c) What is the replacement rate G required to maintain 12,000 gal of water in the pool?

Let $V(t)$ be the number of gallons of water in the pool at time t days.

Then units on $V' = \frac{dV}{dt}$ are Gallons/Day

Now $V' =$ rate in - rate out

Rate In: G.

Rate Out?

0.0005V.

$$\text{So } V' = G - 0.0005V = G - \frac{V}{2000}$$

(a) Equilibrium Volume is: $V_e = 2000G$

(b) If pool volume is initially 1% above its equilibrium, find an expression for $V(t)$.

The Initial Value Problem is:

$$V' + \frac{V}{2000} = G, V(0) = 1.01V_e = (1.01)(2000)G = 2020G$$

$$V' + \frac{V}{2000} = G, V(0) = 2020G$$

The DE is linear with integrating factor $\mu = e^{t/2000}$.
The general solution is

$$V(t) = 2000G + Ce^{-t/2000}$$

With $V(0) = 2020G$, we get $C = 20G$

and solution is $V = 2000G + 20Ge^{-t/2000}$

What is the replacement rate G required to maintain 12,000 gallons of water in the pool?
$12000 = 2000G$ so $G = 6$ gallons per day.

13. A recent college graduate borrows \$100,000 at an interest rate of 9% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of $800(1 + t/120)$, where t is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Distances will be measured in **feet**, time in **seconds**
With $g = 32$ ft/sec/sec, $mg = 180$, so $m = \frac{180}{32} = \frac{45}{8}$
and $\frac{1}{m} = \frac{8}{45}$.

(a) Measure the positive direction downward.
In First 10 seconds, air resistance force is).75v
Newton's Law of Motion: (mass)(acceleration) = sum of forces

$$m\frac{dv}{dt} = -0.75v + mg\frac{dv}{dt} = -0.75\frac{v}{m} + g\frac{dv}{dt} = -\frac{2}{15}v + 32$$

$$\text{Thus } v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Part (a) Continued

$$v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Solution is $v(t) = 240(1 - e^{-\frac{2}{15}t})$

At 10 seconds:

$$v(10) = 240(1 - e^{-\frac{2}{15}(10)}) = 240(1 - e^{-\frac{4}{3}}) \sim 176.7 ft/sec$$

This is the speed of the skydiver when the parachute opens

(b) Find the distance fallen before the parachute opens.
We want $s(10)$ where $s(t)$ is the number of feet fallen after t seconds.
Note: $s(0) = 0$ and $v(t) = s'(t)$ so $s(t) = \int v'(t)dt$
$s(t) = \int 240 - 240e^{-\frac{2}{15}t} = 240t + 1800e^{-\frac{2}{15}t} + C$
Since $s(0) = 0$, we have $C = -1800$.

$$\text{Thus } s(t) = 240t + 1800e^{-\frac{2}{15}t} - 1800$$

Then $s(10) = 600 + 1800e^{-\frac{4}{3}} \sim 1074.47$ feet

(c) What is the limiting velocity v_L **after** the parachute opens?
"Force of air resistance is $12|v|$ when the parachute is open."

$$m\frac{dv}{dt} = -12v + mg \text{ or } v' = -\frac{12}{m}v + 32 = -\frac{32}{15}v + 32$$

Solution of differential equation is $v(t) = 15 + Ce^{-\frac{32}{15}t}$
Limiting velocity will be 15 ft/sec regardless of the value of C.
If we start the timer again at $t = 0$ when the parachute is open, then $v(0) = 176.7$
This yields a value for C of $C = 161.7$.
Hence $v(t) = 15 + 161.7e^{-\frac{32}{15}t}$

(d) Determine how long the skydiver is in the air after the parachute opens.
Since we know the velocity, we can find the distance $s(t)$ the skydiver has fallen by integrating $v(t)$.

$$s(t) = \int v(t)dt = \int 15 + 161.7e^{-\frac{32}{15}t}dt = 15t - 75.8e^{-\frac{32}{15}t} + D$$

Now use $s(0) = 1074.5$ to find $D = 1150.3$.
The skydiver will hit the ground at time T where $s(T) = 5000$.
The answer is $T = 256.6$ seconds.