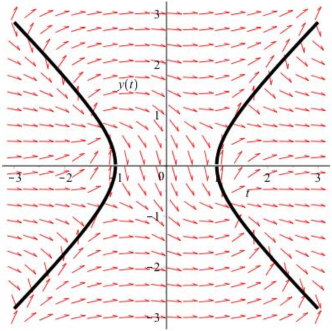
MATH 226: Differential Equations



Class 9: February 28, 2025

- Notes on Assignment 5
- Revised Assignment 6
- Two Fundamental Existence and Uniqueness Theorems

Announcements

First Team Project
Due: Next Friday

Exam 1: Wednesday, March 12

Today's Topics

Existence and Uniqueness Theorems for Nonlinear Differential Equations

Qualitative Analysis of Single Species Population Dynamics Autonomous Models

25. A skydiver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5,000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is $0.75|v|$ when the parachute is closed and $12|v|$ when the parachute is open, where the velocity v is measured in feet per second.

(a) Find the speed of the skydiver when the parachute opens.
(b) Find the distance fallen before the parachute opens.
(c) What is the limiting velocity v_L after the parachute opens?
(d) Determine how long the skydiver is in the air after the parachute opens.
(e) Plot the graph of velocity versus time from the beginning of the fall until the skydiver reaches the ground.

Distances will be measured in **feet**, time in **seconds**
With $g = 32$ ft/sec/sec, $mg = 180$, so $m = \frac{180}{32} = \frac{45}{8}$
and $\frac{1}{m} = \frac{8}{45}$.

(a) Measure the positive direction downward.
In First 10 seconds, air resistance force is).75v
Newton's Law of Motion: (mass)(acceleration) = sum of forces

$$m\frac{dv}{dt} = -0.75v + mg\frac{dv}{dt} = -0.75\frac{v}{m} + g\frac{dv}{dt} = -\frac{2}{15}v + 32$$

$$\text{Thus } v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Part (a) Continued

$$v' + \frac{2}{15}v = 32 \text{ with } v(0) = 0$$

Solution is $v(t) = 240(1 - e^{-\frac{2}{15}t})$

At 10 seconds:

$$v(10) = 240(1 - e^{-\frac{2}{15}(10)}) = 240(1 - e^{-\frac{4}{3}}) \sim 176.7 ft/sec$$

This is the speed of the skydiver when the parachute opens

(b) Find the distance fallen before the parachute opens.
We want $s(10)$ where $s(t)$ is the number of feet fallen after t seconds.
Note: $s(0) = 0$ and $v(t) = s'(t)$ so $s(t) = \int v'(t)dt$
$s(t) = \int 240 - 240e^{-\frac{2}{15}t} = 240t + 1800e^{-\frac{2}{15}t} + C$
Since $s(0) = 0$, we have $C = -1800$.

$$\text{Thus } s(t) = 240t + 1800e^{-\frac{2}{15}t} - 1800$$

Then $s(10) = 600 + 1800e^{-\frac{4}{3}} \sim 1074.47$ feet

(c) What is the limiting velocity v_L **after** the parachute opens?
"Force of air resistance is $12|v|$ when the parachute is open."

$$m\frac{dv}{dt} = -12v + mg \text{ or } v' = -\frac{12}{m}v + 32 = -\frac{32}{15}v + 32$$

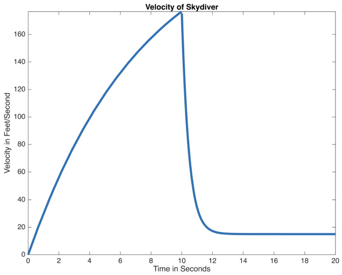
Solution of differential equation is $v(t) = 15 + Ce^{-\frac{32}{15}t}$
Limiting velocity will be 15 ft/sec regardless of the value of C.
If we start the timer again at $t = 0$ when the parachute is open, then $v(0) = 176.7$
This yields a value for C of $C = 161.7$.
Hence $v(t) = 15 + 161.7e^{-\frac{32}{15}t}$

(d) Determine how long the skydiver is in the air after the parachute opens.
Since we know the velocity, we can find the distance $s(t)$ the skydiver has fallen by integrating $v(t)$.

$$s(t) = \int v(t)dt = \int 15 + 161.7e^{-\frac{32}{15}t}dt = 15t - 75.8e^{-\frac{32}{15}t} + D$$

Now use $s(0) = 1074.5$ to find $D = 1150.3$.
The skydiver will hit the ground at time T where $s(T) = 5000$.
The answer is $T = 256.6$ seconds.

(e) Plot the graph of velocity versus time from beginning of the fall until skydiver reaches the ground.



Here are *MATLAB* commands that would generate this graph:

```
syms t
y = piecewise(t < 10,240 - 240*exp(-(2/15)*t),10 < t<=
20,15 +161.7*exp((-32/15)* (t-10)))
fplot(t,y,[0 20], 'LineWidth', 3)
xlabel('Time in Seconds')
ylabel('Velocity in Feet/Second')
title('Velocity of Skydiver')
```

Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

Notes: Theorem 2.4.1 pertains to **LINEAR** systems

Find longest open interval I containing t_o on which $p(t)$ **and** $g(t)$ are continuous.

You need not find the solution itself to find its interval of definition.

Solutions with different initial conditions do not intersect over their intervals of definition.

B.Existence and Uniqueness Theorems

Theorem 2.4.1 If the functions p and g are continuous on an open interval $I : \alpha < t < \beta$ containing the point $t = t_0$, then there exists a unique function $y = \phi(t)$ that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for each t in I, and that also satisfies the initial condition

$$y(t_0) = y_0,$$

where y_0 is an arbitrary prescribed initial value.

PROOF: Follow construction of the solution

$$\phi(t) = \frac{1}{\mu(t)}[\int_{t_o}^{t} \mu(s)g(s)ds + \mu(t_o)y_o]$$

where

$$\mu(t) = e^{\int p(t)dt}$$

Example

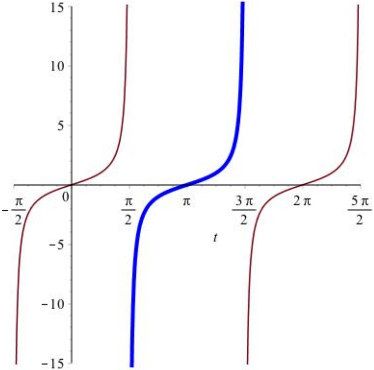
$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

$$g(t) = \sin t$$

is continuous for all t

$$p(t) = \tan t$$

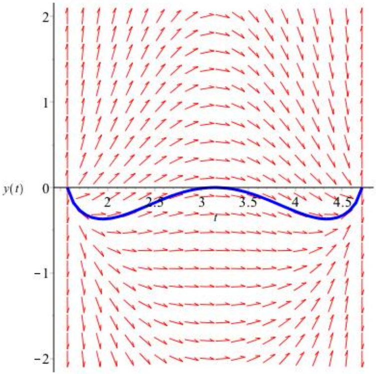
is continuous on $(\pi/2, 3\pi/2)$.



$$y' + (\tan t)y = \sin t, y(\pi) = 0$$

Has Solution

$$y = -\cos t \ln|\cos t|$$



Theorem on Nonlinear Differential Equations

$$y' = f(t, y) \text{ with } y(t_o) = y_o$$

Suppose there is an open rectangle R in (t, y)-plane and (t_o, y_o) is in R.

IF both f and $\partial f/\partial y$ are continuous throughout R,

THEN there is some open interval I centered around t_o and a unique solution $y = \phi(t)$ of the differential equation valid on I with $\phi(t_o) = y_o$

Theorem 2.4.2 Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y), \qquad y(t_0) = y_0.$$

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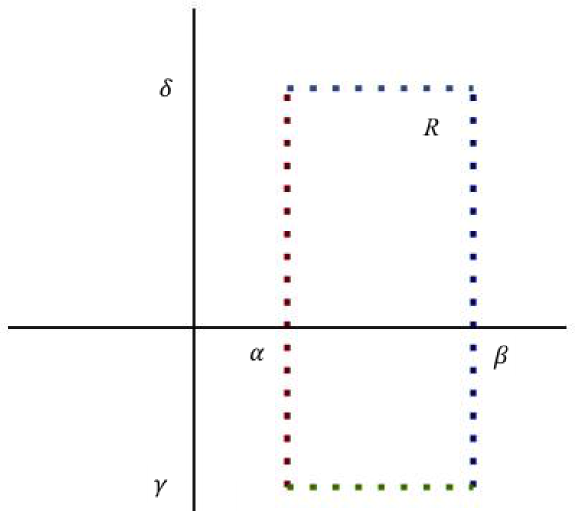

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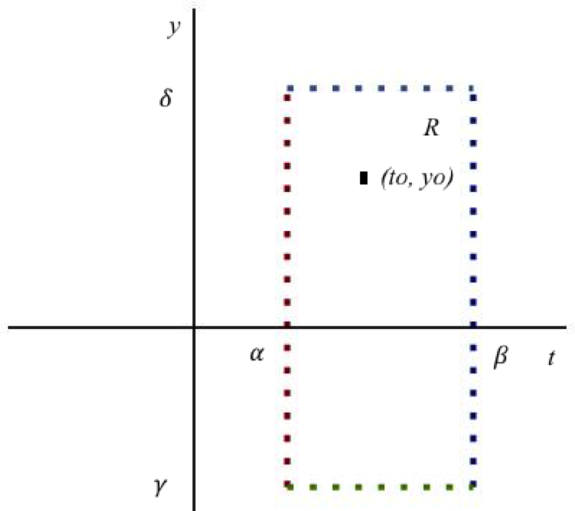

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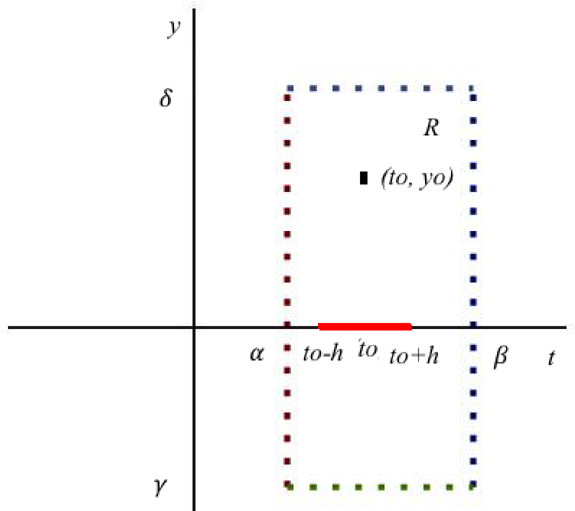


Example

$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

$$f(t, y) = \frac{\ln|ty|}{1 - t^2 + y^2}, \ \frac{\partial f}{\partial y} = \frac{(1 - t^2 + y^2)(1/y) - \ln|ty|2y}{(1 - t^2 + y^2)^2}$$

Fails to be continuous when
$t = 0$ or $y = 0$ or $1 - t^2 + y^2 = 0$.

Example

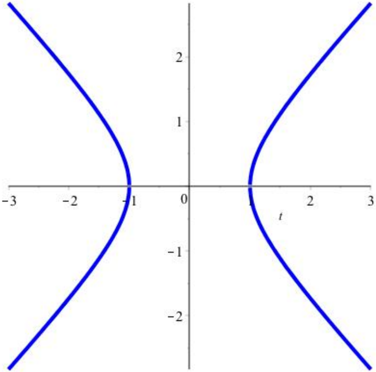
$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$

Fails to be continuous when $1 - t^2 + y^2 = 0$ or $y^2 = t^2 - 1$

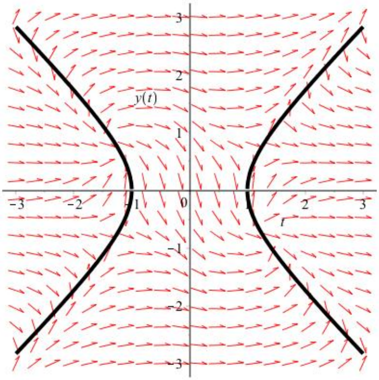
Examine $y^2 = t^2 - 1$.

Must have $t^2 - 1 \geq 0$ so $t^2 \geq 1$

So $t \geq 1$ or $t \leq -1$



$$y' = \frac{\ln|ty|}{1 - t^2 + y^2}$$



Example

$$y' = y^2 \text{ with } y(0) = 3$$

$f(t, y) = y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous.

Separate Variables

$$y^{-2} y' = 1$$

$$-y^{-1} = t + C$$

Find C: $-3^{-1} = 0 + C$ so $C = -\frac{1}{3}$

$$-\frac{1}{y} = t - \frac{1}{3}$$

$$\frac{1}{y} = -t + \frac{1}{3} = \frac{-3t + 1}{3}$$

$$y = \frac{3}{1 - 3t}$$

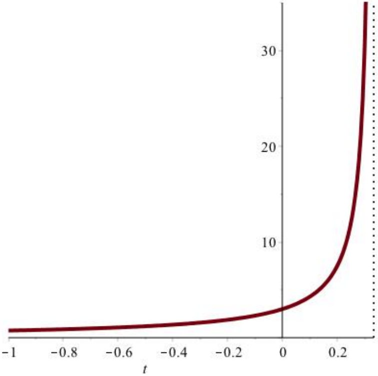
Example

$$y' = y^2 \text{ with } y(0) = 3$$

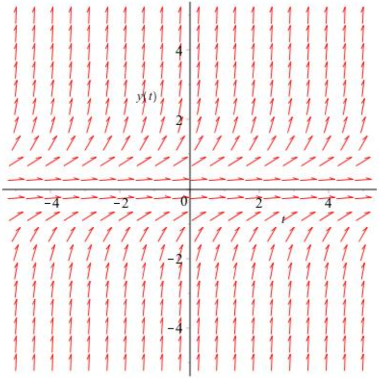
Has Solution

$$y = \frac{3}{1 - 3t}$$

Solution valid on $(-\infty, \frac{1}{3})$



$$y' = y^2$$

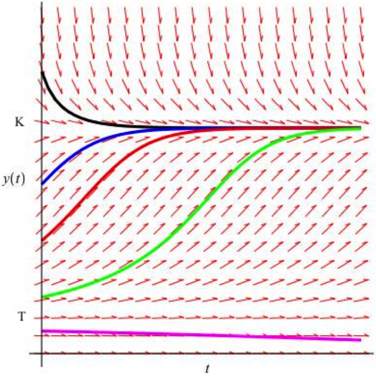


Qualitative Analysis of Single Species Population Dynamics
Autonomous Models
y' = f(y)

Logistic Growth With Threshold

$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

K is Carrying Capacity and T is Threshold.

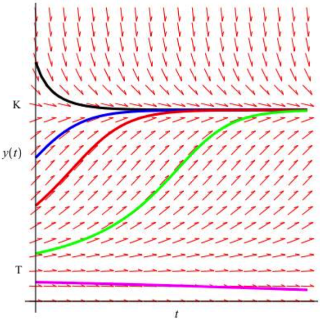


$$y' = ry(1 - \frac{y}{K})(\frac{y}{T} - 1), 0 < T < K$$

$$y' = 0 \text{ when } y = 0, y = K, y = T$$

$$\begin{array}{ll} y' < 0: & 0 < y < T \\ y' > 0 & T < y < K \\ y' < 0: & y > K \end{array}$$

$$y'' = [-\frac{3}{KT}y^2 + (\frac{2}{T} + \frac{2}{K})y - 1]y'$$



Preview of Coming Attractions:

Systems of First Order Differential Equations

$$\frac{dx}{dt} = ax(t) + by(t) + f(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t) + g(t)$$

Review Linear Algebra