

1. *Mice and Owls Revisited: A Linear Predator Prey Model.* We consider a simple two species ecosystem with mice and owls. The owls only eat mice so if there are no mice, the owl population will decay to zero. The owl is also the only predator for the mice so if there are no owls, the mice will thrive. On the other hand, the presence of owls is bad for the mice while the presence of mice is good for the owls. Here is a simple model of the Mice and Owl populations, letting x be the population of mice at time t and y the owl population at time t . Suppose our model is the system of differential equations

$$x' = \frac{1}{10}x - \frac{1}{5}y, \quad y' = \frac{3}{10}x - \frac{2}{5}y$$

- Formulate this model as a linear homogenous system in vector form $\mathbf{x}' = A\mathbf{x}$ by carefully describing the entries of $\mathbf{x}$ and the matrix A .
 - Find the characteristic polynomial for A . [You may convert the fractions to decimals if you prefer to work with decimals].
 - Determine the eigenvalues of A .
 - For this model, what will happen to the mice and owl populations in the long term. [It may or may not be helpful to know that $\sqrt{2} \sim 1.414$, $\sqrt{3} \sim 1.732$, $\sqrt{5} \sim 2.236$ and $\sqrt{6} \sim 2.449$].
2. The eigenvalues for the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$ are 2 and -3.
- For each eigenvalue, find a corresponding eigenvector at least one of whose components is 1.
 - Find and classify the critical point of the system $\mathbf{x}' = A\mathbf{x}$ for this matrix A ; for example, is unstable or asymptotically stable? A node or a center?
3. Consider the homogeneous system $\mathbf{x}' = A\mathbf{x}$ where A is the matrix $\begin{pmatrix} 4 & -3 \\ 8 & -6 \end{pmatrix}$
- Verify that $\mathbf{v} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is an eigenvector of this matrix. Without computing a characteristic polynomial, determine the corresponding eigenvalue.
 - Given that $\lambda = -2$ is another eigenvalue which has $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ as an eigenvector, what is the general solution of the system of differential equations?
 - Find the particular solution which has $\mathbf{x}(0) = \begin{pmatrix} 14 \\ 22 \end{pmatrix}$
4. Let A be the matrix $\begin{pmatrix} -1 & -1/2 \\ 2 & -3 \end{pmatrix}$
- Show that A has an eigenvalue $\lambda = -2$ of algebraic multiplicity 2.
 - Show that $\lambda = -2$ has geometric multiplicity 1.
 - Find two linearly independent solutions of the system $\mathbf{x}' = A\mathbf{x}$ for this matrix A .

5. Suppose Xavier is in love with Yetta, but Yetta is a fickle lover. The more Xavier loves her, the more she dislikes him – but when he loses interest in her, her feelings for him warm up. On the other hand, Xavier reacts to her: when she loves him, his love for Yetta grows and when she loses interest, he also loses interest. Let $x(t)$ = Xavier's feelings for Yetta at time t and $y(t)$ = Yetta's feelings for Xavier at time t where positive and negative values of the variables x and y denote love and dislike, respectively. The exact units by which these variables can be measured will be left to the imagination of the reader.

Our fundamental assumption is: *the change in one person's feelings at any time is directly proportional to the other person's feelings at the same time.* Suppose we model the relationship between Xavier and Yetta by the system of differential equations $dx/dt = ay$, $dy/dt = -bx$ where a and b are positive constants. Determine the general solution of this system and discuss its long term behavior.

6. Let A be a 50 by 50 matrix and $\mathbf{0}$ the function which is the constant zero function; that is, $\mathbf{0}(t)$ is the vector of all zero's for every t .
- Show that $\mathbf{0}$ is an equilibrium point for the system of differential equations $\mathbf{x}' = A\mathbf{x}$.
 - Prove that if A is invertible, then $\mathbf{0}$ is the only equilibrium point for $\mathbf{x}' = A\mathbf{x}$.
 - Show that if A is not invertible, then there is a nonzero equilibrium vector for $\mathbf{x}' = A\mathbf{x}$.
 - Prove that 0 is an eigenvalue for A if and only if A is not invertible.