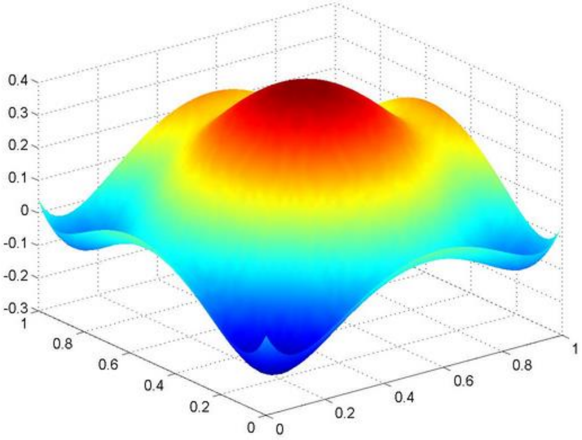
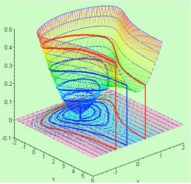


It is a curious historical fact that modern quantum mechanics began with two quite different mathematical formulations: the **differential equations** of Schroedinger and the **matrix algebra** of Heisenberg. The two apparently dissimilar approaches were proved to be mathematically equivalent. *Richard Feynman*

MATH 226: Differential Equations

Michael Olinick
Middlebury College



February 10, 2025

What Are Differential Equations?

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- **Orindary** differential equation (ODE): unkown function has *only* one independent variable.

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- **Partial** differential equation (PDE): unknown function has more than one independent variable.

$$u = u(x, y), u_{xx} + u_{yy} = 0 (or \Delta u = 0)$$

What Are Differential Equations?

The **order** of a differential equation is the order of the highest derivative appearing in the equation.

All differential equations can be written in the form

$$F(\text{independent variable, dependent variable, variable and derivatives}) = 0$$

where all derivatives up to the highest power in the equation are variables in F.

$$\frac{dy}{dt} = ky, \frac{dy}{dt} - ky = 0, F(t, y, \frac{dy}{dt}) = \frac{dy}{dt} - ky$$

$$u_{xx} + u_{yy} = 0, F(x, y, u_x, u_y, u_{xx}, u_{yy}) = u_{xx} + u_{yy}$$

What is a Solution To a Differential Equation?

Give the ODE

$$F(t, y, y', y'', ..., y^{(n)}) = 0$$

a **solution** is a function $y = \phi(t)$ satisfying the equation for all t in some open interval I:

1. ϕ is n times differentiable in I.
2. ϕ satisfies the equation for all t in I.

We say that $y = \phi(t)$ is **a solution to the differential equation** on I.

A Differential Equation and Solution

$$Equation : \frac{dy}{dt} = 12y$$

$$Solution : \phi(t) = 9e^{12t}$$

$$Check : \phi'(t) = 9(12e^{12t}) = 12(9e^{12t}) = 12\phi(t)$$

Why Do We Care About Differential Equations?

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Born:December 17, 1842, Nordfjordeid, Norway
Died: February 18, 1899, Oslo, Norway
[MacTutor Biography]

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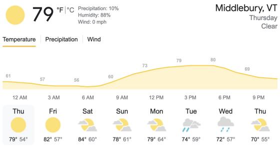
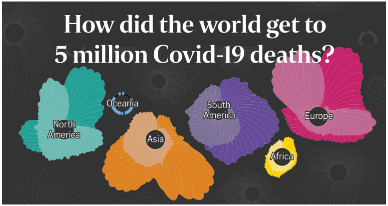
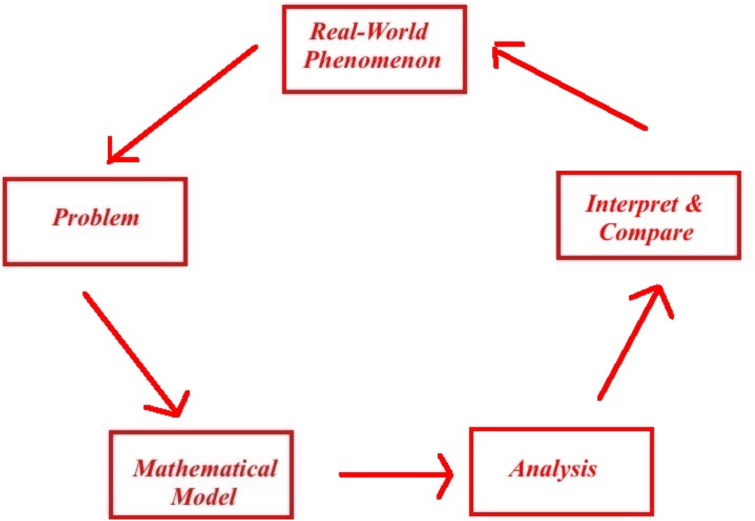
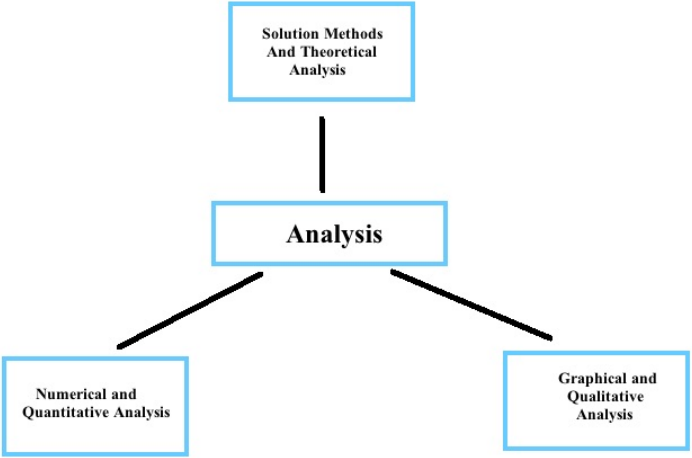




Mathematical Modeling



Analyzing Differential Equations



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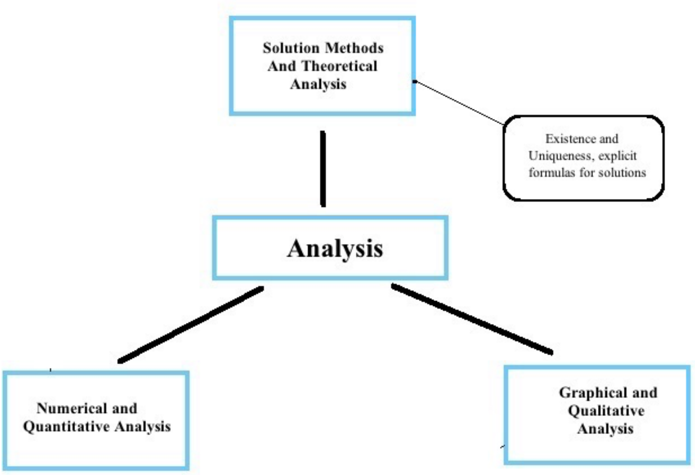

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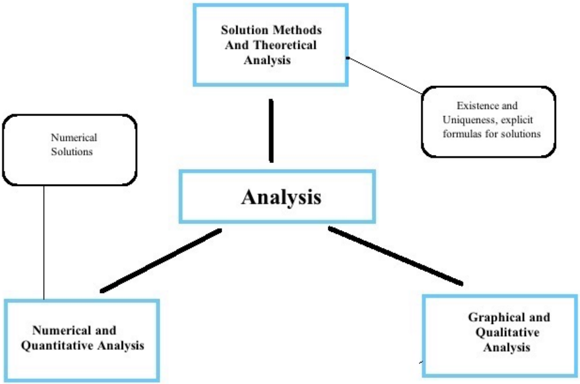

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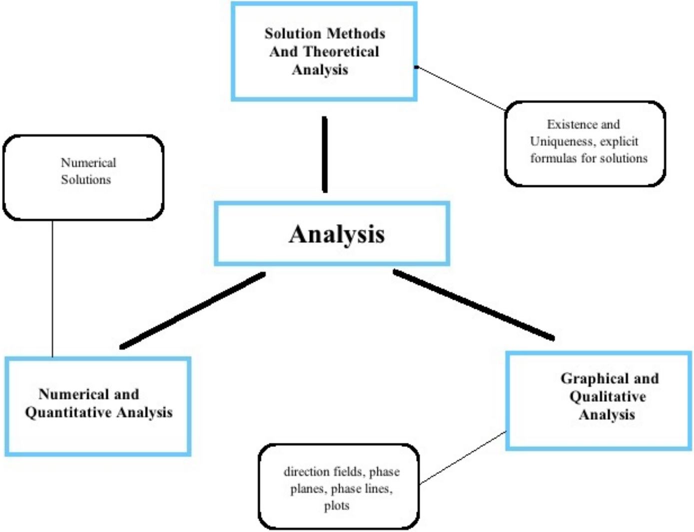

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An equation that gives some explicit information about the **derivative** of a function. but not about the function itself.

Goal: Solve the equation to find the underlying function.

Example 1

$$y' = 2x, \frac{dy}{dx} = 2x, f'(x) = 2x$$

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Note: We can always check our proposed answer.
Can there be any other solution?

Example 2: Generalize Example 1

$$y' = g(x), \frac{dy}{dx} = g(x), f'(x) = g(x)$$

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$$y = f(x) = \int g(x)dx$$

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The Integration (or Antiderivative) Problem

Techniques:

Substitution = Change of Variable

Integration By Parts

Partial Fraction Decomposition

Where Do Differential Equations Arise?
Derivative is Measure of Rate of Change

Physical laws may give us information on how things evolve over time.

Derivatives will be with respect to time.

Notation:

Independent Variable: t, x

Dependent Variable: y, P, u

Example 3

$$P'(t) = 3P(t) \text{ with } P(0) = 100$$

Initial Value Problem

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Initial Value Problem

Applications:

Colony of Bacteria

Money Compounded Continuously

Human Population with Constant Per Capita Growth Rate

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Qualitative Analysis

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P' is positive so P is increasing.

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$$P'' = (P')' = (3P)' = 3P' = 3 \times 3P = 9P$$

So $P'' > 0$ and hence graph of P is increasing and concave up.

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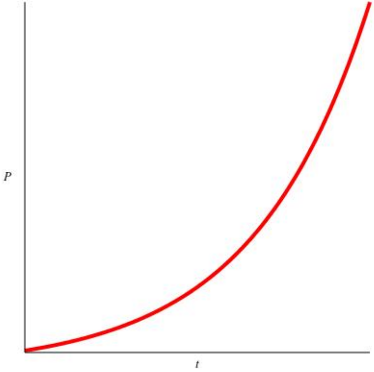

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Analytic Solution

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P is unbounded

$$\lim_{t\to\infty} P(t) = +\infty$$

How about $P' = -3P, P(0) = 100$?

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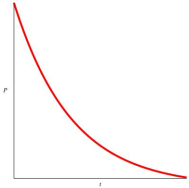
Application: Radioactive Decay

P' is initially negative, so P is decreasing

$$P'' = (P')' = (-3P)' = -3P' = -3(-3)P = 9P > 0$$

Hence P is decreasing and graph is concave up



Analytic Solution (Go Through Similar Steps)

$$P(t) = 100e^{-3t}$$

Observe: $P(t) > 0$ for all t

$$\lim_{t\to\infty} P(t) = 0$$

MORE GENERALLY:

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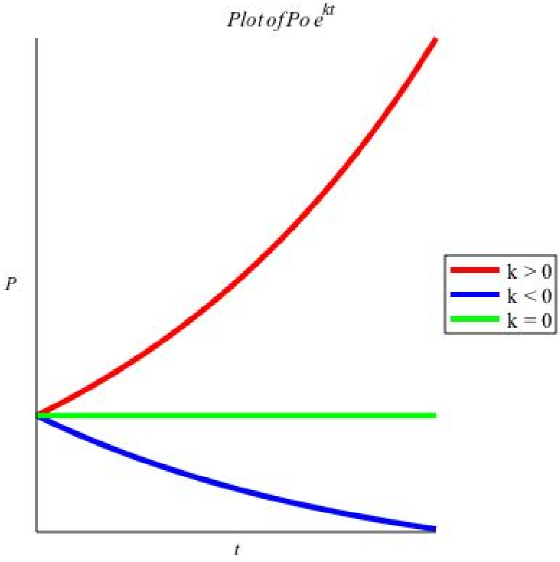
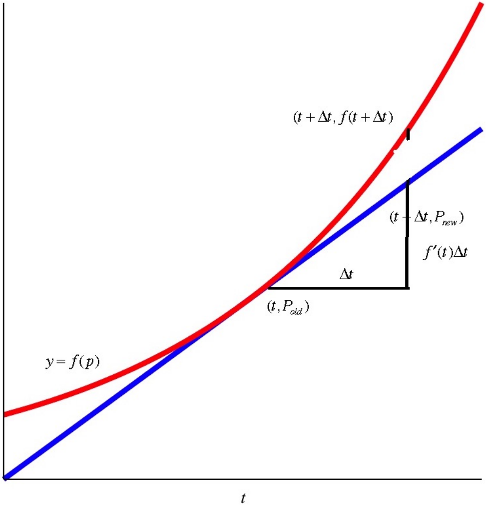

Plot of Po e^kt
P
t
k > 0
k < 0
k = 0

Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$



Euler's Method For "Solving" Numerically $P'(t) = kP(t)$

$$P_{new} = P_{old} + k * P_{old} * \Delta t$$

Example: $k = .04, P(0) = 1000$

Time	Approximate	Exact
0.	1000.00	1000.00
0.1000000000	1004.00000	1004.008011
0.2000000000	1008.016000	1008.032086
0.3000000000	1012.048064	1012.072289
0.4000000000	1016.096256	1016.128685
0.5000000000	1020.160641	1020.201340
0.6000000000	1024.241284	1024.290318
0.7000000000	1028.338249	1028.395684
0.8000000000	1032.451602	1032.517505
0.9000000000	1036.581408	1036.655846
1.000000000	1040.727734	1040.810774

Generalizations

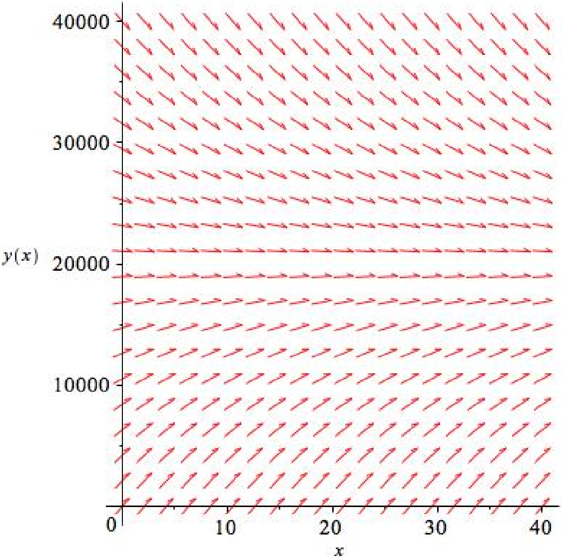
1. Population with immigration and/or emigration
2. Forest Management
3. Fishery Management
4. Lake Champlain Pollution
5. Anesthetic
6. Alcohol/Drug

$$P' = aP + b$$

Key Terms From Chapter 1

Independent Variable
Dependent Variable
Parameter
Solution
Equilibrium Solution
Integral Curves
Autonomous Differential Equation
Critical Point = Fixed Point = Stationary Point
Phase Line
One - Dimensional Phase Portrait
Asymptotically Stable
Unstable
Semistable Attractor = Sink
Repeller = Source
Linearization About An Equilibrium
Direction Field

Direction Field for $P' = -.05P + 1000$



Some Integral Curves For $P' = -.05P + 1000$