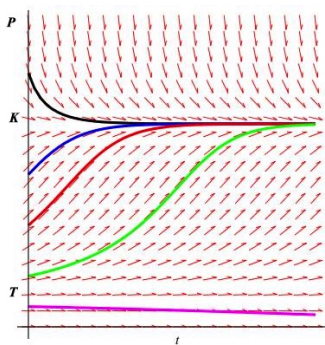


# MATH 226 Differential Equations



Class 10: March 3, 2025



## ► Notes on Assignment 6

# Announcements

First Team Project  
Due: Friday

**Exam 1: Wednesday, March 12**

## Qualitative Analysis of Single Species Population Dynamics

### Autonomous Models

$$y' = f(y) \text{ or } P' = f(P)$$

Logistic Growth With Threshold

$$P' = rP\left(1 - \frac{P}{K}\right)\left(\frac{P}{T} - 1\right), 0 < T < K$$

$K$  is Carrying Capacity and  $T$  is Threshold.

Development and Qualitative Analysis  
of Logistic Model of Population Growth  
With a Threshold

## Model I: Simple Exponential Growth

$$P' = rP, r > 0$$

The parameter  $r$  is called the *Intrinsic Growth Rate*

Model asserts that population grows at a constant percentage rate.

Equivalent to *Constant Per Capita Growth*:

$$\frac{P'}{P} = r$$

Predicts: Population will grow more and more rapidly over time  
and be an **unbounded** function of  $t$ .

Model is often realistic in early stages of growth, but fails in long  
term as we have only a finite amount of resources.

Solution is

$$P(t) = P(0)e^{rt}$$

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Model is often realistic in early stages of growth, but fails in long  
term as we have only a finite amount of resources.

## Model II: Logistic Growth

Population growth must slow down as population becomes large.

There is a *Carrying Capacity*, the largest population the environment can sustain.

$$P' = rP \left( 1 - \frac{P}{K} \right), r, K > 0$$

When  $P$  is small compared to  $K$ ,  $1 - \frac{P}{K}$  is close to 1 so we have near exponential growth for small populations. As  $P$  approaches  $K$ ,  $P'$  gets closer to 0.

$P'$  is positive for  $0 < P < K$  and negative if  $P > K$ .

See our text for more analysis.

### Model III: Logistic Growth with a Threshold

If the population is very small, then it may difficult to find a mating partner and population will actually decrease; perhaps even move toward extinction

$$P' = rP \left(1 - \frac{P}{K}\right) \left(\frac{P}{T} - 1\right), 0 < T < K$$

If  $P$  is smaller than  $T$ , then  $\frac{P}{T} - 1$  will be negative and  $P$  will decrease.

$T$  is called a *threshold* and typically is at least a couple of orders of magnitude smaller than  $K$

Think of  $K = 1000$ ,  $T = 10$  for an example.

When is  $P$  Increasing and When is it Decreasing?

Analyze Sign of  $P'$ . Since  $P$  is nonnegative, the sign of  $P'$  is the same as the sign of the product  $(1 - \frac{P}{K})(\frac{P}{T} - 1)$ .

The graph of this product is a downward opening parabola so it is positive only when  $T < P < K$ .

Thus we have

|             |          |                |
|-------------|----------|----------------|
| $0 < P < T$ | $P' < 0$ | $P$ decreasing |
| $T < P < K$ | $P' > 0$ | $P$ increasing |
| $P > K$     | $P' < 0$ | $P$ decreasing |

To Determine Concavity, We Need To Examine Sign of  $P''$

Idea: Multiply out expression for  $P'$  to get a polynomial.

$$P' = rP \left(1 - \frac{P}{K}\right) \left(\frac{P}{T} - 1\right) = r \left(-\frac{P^3}{KT} + P^2 \left(\frac{T+K}{KT}\right) - P\right)$$

Now differentiate term by term, using the Chain Rule:

$$P'' = r \left(\frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1\right) P'$$

We know the sign of  $r$  (+) and the sign of  $P'$  for all values of  $P$ .

We need to find the sign of

$$Q(P) = \frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1$$

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The graph of  $Q$  as a function of  $P$  is a downward opening parabola. It has two roots  $a < b$  so  $Q(P)$  is positive for  $a < P < b$  and negative for other values of  $P$ .

To find  $a$  and  $b$ , we need to solve  $Q(P) = 0$

$$\frac{-3P^2}{KT} + 2\frac{T+K}{KT}P - 1 = 0$$

Multiply through by  $KT$  to clear fractions:

$$-3P^2 + 2(T+K)P - KT = 0$$

Apply Quadratic Formula to

$$-3P^2 + 2(T + K)P - KT = 0$$

$$P = \frac{-2(T + K) \pm \sqrt{4(T + K)^2 - 4(3)KT}}{-6}$$

which simplifies to

$$P = \frac{(T + K) \pm \sqrt{(T + K)^2 - 3KT}}{3}.$$

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Root  $b$ : Take Positive Square Root: First observe that  $3KT$  is relatively very small compared to  $(T + K)^2$

( For  $T = 10, K = 1000$ , we have  $3KT = 30,000$  while  $(T + K)^2$  is well above 1,000,000)

Thus  $(T + K)^2 - 3KT \approx (T + K)^2$  so

$$\sqrt{(T + K)^2 - 3KT} \approx (T + K)$$

$$\text{and } b \approx \frac{(T+K)+(K+T)}{3} = \frac{2}{3}(K + T) \approx \frac{2}{3}K$$

since  $K$  is much larger than  $T$ .

(For  $T = 10, K = 1000$ , exact solution is 668.35)

Root  $a$ : Take Negative Square Root:

$$a = \frac{(T + K) - \sqrt{(T + K)^2 - 3KT}}{3}$$

We will be subtracting from  $T + K$  a number slightly smaller than  $T + K$  so we expect the answer to be quite small.

Using the tangent line approximation  $f(x + h) \approx f(x) + f'(x)h$  with the square root function  $f(x) = \sqrt{x}$  we have

$$\sqrt{x + h} \approx \sqrt{x} + \frac{h}{2\sqrt{x}}.$$

Letting  $x = (T + K)^2$  and  $h = -3KT$ , we have

$$\sqrt{(T + K)^2 - 3KT} \approx (T + K) - \frac{3KT}{2(T + K)}$$

$$\text{Thus } a \approx \frac{(T + K) - (T + K) + \frac{3KT}{2(T + K)}}{3} = \frac{KT}{2(T + K)}$$

We have  $a \approx \frac{KT}{2(T+K)}$  but  $T + K \approx K$  so

$$a \approx \frac{KT}{2K} = \frac{T}{2}$$

(For  $T = 10, K = 1000$ , exact solution is 4.99)

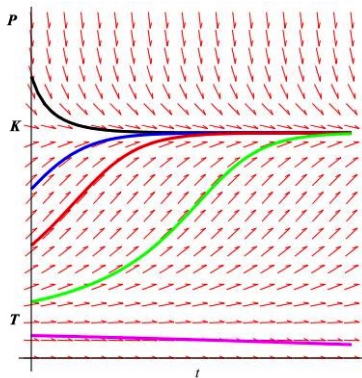
Thus the roots of  $Q(P)$  are approximately  $\frac{1}{2}T$  and  $\frac{2}{3}K$ .  
So  $Q(P)$  is positive for  $P$  roughly between  $\frac{1}{2}T$  and  $\frac{2}{3}K$  and negative elsewhere,

| $P$                    | $Q(P)$   | $P'$     | $P''$    |
|------------------------|----------|----------|----------|
| $0 < P < \frac{T}{2}$  | Negative | Negative | Positive |
| $\frac{T}{2} < P < T$  | Positive | Negative | Negative |
| $T < P < \frac{2}{3}K$ | Positive | Positive | Positive |
| $\frac{2}{3}K < P < K$ | Negative | Positive | Negative |
| $P > K$                | Negative | Negative | Positive |

Summarizing Our Results:

| $P$                    | Graph of $P$             |
|------------------------|--------------------------|
| $0 < P < \frac{T}{2}$  | Decreasing, Concave Up   |
| $\frac{T}{2} < P < T$  | Decreasing, Concave Down |
| $T < P < \frac{2}{3}K$ | Increasing, Concave Up   |
| $\frac{2}{3}K < P < K$ | Increasing, Concave Down |
| $P > K$                | Decreasing, Concave Up   |

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## Preview of Coming Attractions:

### Systems of First Order Differential Equations

$$\frac{dx}{dt} = ax(t) + by(t) + f(t)$$

$$\frac{dy}{dt} = cx(t) + dy(t) + g(t)$$

**Review Linear Algebra**