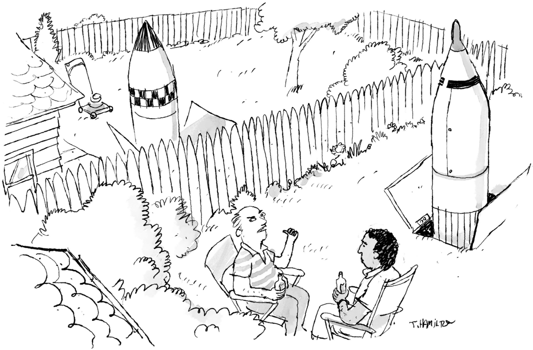
MATH 226: Differential Equations



"You know how it is! We get a warhead, *they* have to get a warhead."

Class 13: March 10, 2025

Notes on Assignment 8

Assignment 9

Graded Project 1

Announcements

- Exam 1
  - Wednesday
  - 7 PM - ? (No Time Limit)
    Warner 104
  - No Calculators, Books, Computers, Smart Phones, etc.
  - One Page of Notes
  - Focus on Material in Chapters 1 and 2

Exam 1 Tips

- Read Overall Directions on Cover Page
- Read Each Problem Statement Carefully
- Pick "Easy" Problem To Do First
- Show All Intermediate Steps
- Include Explanations
- Pay Attention to Units
- Double Check Your Work

Solving $\mathbf{X'} = A\mathbf{X}$ where A is an n by n matrix of constants and $\mathbf{X}$ is is an n by 1 vector of functions.

What We Know So Far:

If λ is an eigenvalue of A and $\vec{v}$ is an associated eigenvector, then

$$e^{\lambda t}\vec{v} \text{ is a solution of } \mathbf{X'} = A\mathbf{X}$$

Theorem: Suppose $\lambda \neq \mu$ are two distinct eigenvalues of a square matrix A with respective eigenvectors $\vec{v}$ and $\vec{w}$; That is,

$A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$

Then $\{\vec{v}, \vec{w}\}$ is a Linearly Independent set of vectors.

Consequently, $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}\}$ is a Linearly Independent set of solutions of $\mathbf{X'} = A\mathbf{X}$

Now let's focus on the 2 by 2 case with initial conditions

Find $\mathbf{X} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ such that $\mathbf{X'} = A\,\mathbf{X}$ and

$$\mathbf{X}(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Claim: There are constants α and β such that $\mathbf{X} = \alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w}$ will do the trick.

Note that $\mathbf{X(0)} = \alpha\vec{v} + \beta\vec{w} = \alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \beta \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

So we need to solve the system

$$\begin{pmatrix} v_1\alpha + w_1\beta \\ v_2\alpha + w_2\beta \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

This system has a unique solution because the coefficient matrix is nonsingular. **WHY?**

$$\begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

has unique solution

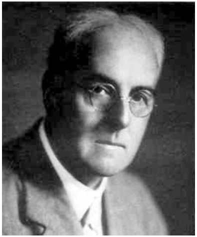
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Exercise Show there are always unique values for α and β for any pair of initial values x_0, y_0 even if the initial time is not zero but some other time t^*.

Today's Topic

Analysis of The Richardson Arms Race Model

Richardson Arms Race Model



Lewis F. Richardson

1881 – 1953

$x(t)$ = **Arms Expenditure of Blue Nation**

$y(t)$ = **Arms Expenditure of Red Nation**

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

where a, b, m, n are positive constants while r and s are constants.

Structure: $\vec{X} = A\vec{X} + \vec{b}$ or $\mathbf{x'} = A\mathbf{x} + \mathbf{b}$

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

where a, b, m, n are positive constants while r and s are constants.

Find stable Lines L and L' where $x' = 0$ and $y' = 0$.

$$L : y = \frac{m}{a}x - \frac{r}{a}$$

$$L' : y = \frac{b}{n}x + \frac{s}{n}$$

Determine the Stable Point (x^*, y^*) where lines **L** and **L'** intersect.

$$ay^* - mx^* + r = 0,\ \ bx^* - ny^* + s = 0$$

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

$$ay^* - mx^* + r = 0, \;\; bx^* - ny^* + s = 0$$

Make Change of Variable $X = x - x^*, \;\; Y = y - y^*$

Then

$$X' = x' = a(Y + y^*) - m(X + x^*) + r = aY - mX + (ay^* - mx^* + r)$$

$$= aY - mX + 0 = aY - mX$$

Similarly, $Y' = bX - nY$

Write system as

$X' = aY - mX$

$Y' = bX - nY$

We transform

$$x' = ay - mx + r$$
$$y' = bx - ny + s$$

a nonhomgeneous system into

$$X' = -mX + aY$$
$$Y' = bX - nY$$

a homogeneous system.

$$\mathbf{X'} = A\,\mathbf{X}$$

where

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix}$$

We can solve by finding eigenvalues and eigenvectors of A

$$\mathbf{X'} = A\,\mathbf{X}$$

where

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix} \text{ has solution}$$

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w}$$

where α and β are arbitrary constants
λ is an eigenvalue of A with associated eigenvector $\vec{v}$ and
$\mu \neq \lambda$ is an eigenvalue of A with associated eigenvector $\vec{w}$.

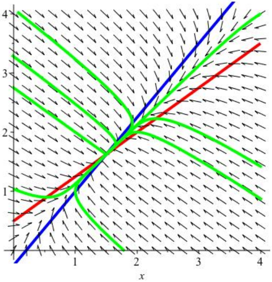
The solution of the original system is then

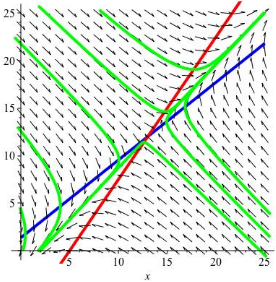
$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w} + \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

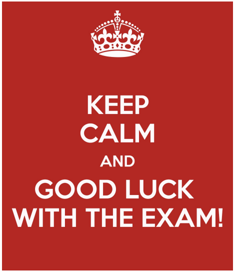
Two Particular Examples:

$x' = -5x + 4y + 1$ $y' = 3x - 4y + 2$	$x' = 11y - 9x - 15$ $y' = 12x - 8y - 60$
$(x^*, y^*) = (\frac{3}{2}, \frac{13}{8})$	$(x^*, y^*) = (13, 12)$
$A = \begin{bmatrix} -5 & 4 \\ 3 & -4 \end{bmatrix}$ $\lambda = -1, \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\mu = -8, \vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$	$A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$ $\lambda = 3, \vec{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$ $\mu = -20, \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\alpha e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{-8t} \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \end{bmatrix}$	$\alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$

$\begin{aligned} x' &= -5x + 4y + 1 \\ y' &= 3x - 4y + 2 \end{aligned}$	$\begin{aligned} x' &= 11y - 9x - 15 \\ y' &= 12x - 8y - 60 \end{aligned}$






KEEP
CALM
AND
GOOD LUCK
WITH THE EXAM!


I KNOW YOU CAN DO IT!
THINK
POSITIVE