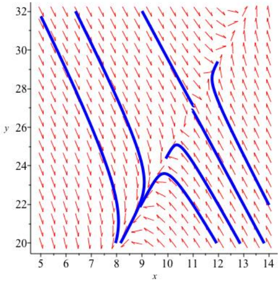
# MATH 226: Differential Equations




Class 14: March 14, 2025

# Announcements

- Exam 1
  - Tonight
  - 7 PM - ? (No Time Limit)
  - Warner 104
  - No Calculators, Books, Smart Phones, etc.
  - Focus on Material in Chapters 1 and 2

Existence and Uniqueness Theorems for Linear **Systems**

**On Course Website:**

Complex Numbers

## Today's Topics

**More Analysis of The Richardson Arms Race Model**

**More About Systems of Two First Order Linear Differential Equations With Constant Coefficients**

## Richardson Arms Race Model

Lewis F. Richardson

*Arms And Insecurity:*

*A Mathematical Study Of The Causes And Origins Of War*

$x(t)$ = **Arms Expenditure of Blue Nation**

$y(t)$ = **Arms Expenditure of Red Nation**

$$x' = ay - mx + r$$

$$y' = bx - ny + s$$

where $a, b, m, n$ are positive constants while $r$ and $s$ are constants.

Structure: $\vec{X} = A\vec{X} + \vec{b}$ or $\mathbf{x'} = A\mathbf{x} + \mathbf{b}$

$$\mathbf{X'} = A\,\mathbf{X} + \mathbf{b}$$

Make Change of Variables

$$X = x - x^*$$

$$Y = y - y^*$$

where $ay^* - mx^* + r = 0, bx^* - ny^* + s = 0$

To Convert To Homogeneous System of Form
$$\mathbf{X'} = A\,\mathbf{X}$$

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where

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix} \text{ has solution}$$

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w}$$

where $\alpha$ and $\beta$ are arbitrary constants
$\lambda$ is an eigenvalue of $A$ with associated eigenvector $\vec{v}$ and
$\mu \neq \lambda$ is an eigenvalue of $A$ with associated eigenvector $\vec{w}$.

The solution of the original system is then

$$\alpha e^{\lambda t}\vec{v} + \beta e^{\mu t}\vec{w} + \begin{bmatrix} x^* \\ y^* \end{bmatrix}$$

Two Particular Examples:

| | |
|---|---|
| $x' = -5x + 4y + 1$<br>$y' = 3x - 4y + 2$ | $x' = 11y - 9x - 15$<br>$y' = 12x - 8y - 60$ |
| $(x^*, y^*) = (\frac{3}{2}, \frac{13}{8})$ | $(x^*, y^*) = (13, 12)$ |
| $A = \begin{bmatrix} -5 & 4 \\ 3 & -4 \end{bmatrix}$<br>$\lambda = -1, \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$<br>$\mu = -8, \vec{w} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ | $A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$<br>$\lambda = 3, \vec{v} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$<br>$\mu = -20, \vec{w} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ |
| $\alpha e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta e^{-8t} \begin{bmatrix} -4 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ \frac{13}{8} \end{bmatrix}$ | $\alpha e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + \beta e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 13 \\ 12 \end{bmatrix}$ |

## *Existence and Uniqueness Theorems for Linear Systems*

***Theorem 2.4.1***: If $p(t)$ and $g(t)$ are continuous functions on an open interval $I$ containing the point $t = t_o$ and $y_o$ is any prescribed initial value, then there exists a unique solution $y = \phi(t)$ of the differential equation that satisfies the differential equation

$$y' + p(t)y = g(t)$$

for all $t$ in $I$ with $\phi(t_o) = y_o$.

***Theorem 3.2.1***: If $P(t)$ is an $n \times n$ matrix and $\mathbf{g}(t)$ is an $n \times 1$ vector whose entries are continuous on an open interval $I$ containing the point $t_o$ and $\mathbf{y_o}$ is any prescribed initial value, then there is a unique solution $\mathbf{y} = \mathbf{\Phi}(t)$ of the system of differential equations

$$\boldsymbol{X}' = P(t)\boldsymbol{X} + \mathbf{g}(t)$$

for all $t$ in I with $\mathbf{\Phi}(t_o) = \mathbf{y_o}$.

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What Does This Theorem Say in the case $P(t)$ is an $n \times n$ matrix of **constants** and $\mathbf{g(t)}$ is identically 0?

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for all $t$ in I with $\mathbf{\Phi}(t_o) = \mathbf{y_0}$.

What Does This Theorem Say in the case $P(t)$ is an $n \times n$ matrix of **constants** and $\mathbf{g(t)}$ is identically 0?

**There is a unique solution valid for all real numbers!**

# Focus on Linear Homogeneous System with Constant Coefficients

$$\mathbf{X'} = A\,\mathbf{X}$$

where $A$ is a $2 \times 2$ matrix.

Begin with Earlier Example

$$x' = -9x + 11y$$
$$y' = 12x - 8y$$

$$A = \begin{bmatrix} -9 & 11 \\ 12 & -8 \end{bmatrix}$$

Two solutions to the homogeneous system are

$$e^{3t}\vec{v} \text{ and } e^{-20t}\vec{w}$$

$$e^{3t}\begin{bmatrix}11\\12\end{bmatrix} \text{ and } e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

Then $C_1 e^{3t}\begin{bmatrix}11\\12\end{bmatrix} + C_2 e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$ is a solution for any constants $C_1$ and $C_2$.

Now suppose $\Phi(t)$ is any solution to the system with $\Phi(0) = \begin{bmatrix}x_0\\y_0\end{bmatrix}$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t) = C_1 e^{3t}\begin{bmatrix}11\\12\end{bmatrix} + C_2 e^{-20t}\begin{bmatrix}1\\-1\end{bmatrix}$$

CLAIM: We can find $C1$ and $C2$ so that

$$\Phi(t) = C_1 e^{3t} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

NEED: Agreement at $t = 0$:

$$C_1 e^{3\times 0} \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 e^{-20\times 0} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$C_1 \begin{bmatrix} 11 \\ 12 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$11C_1 + 12C2 = x_0$$
$$12C_1 - 1C_2 = y_0$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

has a solution for all $x_0, y_0$ exactly when the coefficient matrix
$M = \begin{bmatrix} 11 & 1 \\ 12 & -1 \end{bmatrix}$ is invertible
and this happens if and only the columns of the coefficient matrix are a linearly independent set of vectors.
But the columns are $\vec{v}$ and $\vec{w}$ which are eigenvectors belonging to distinct eigenvalues
so they do form a linearly independent set.

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

The solution will be

$$\begin{bmatrix} C1 \\ C2 \end{bmatrix} = M^{-1} \begin{bmatrix} x_0 \\ y_o \end{bmatrix} = \frac{-1}{23} \begin{bmatrix} -1 & -1 \\ -12 & 11 \end{bmatrix} \begin{bmatrix} x_0 \\ y_o \end{bmatrix}$$

$$\text{Thus } \begin{bmatrix} C1 \\ C2 \end{bmatrix} = \begin{bmatrix} \frac{x_0+y_0}{23} \\ \frac{12x_0-11y_0}{23} \end{bmatrix}.$$

We have found one solution of the homogeneous system that

- Agrees with $\Phi$ and $t = 0$ and
- Is a linear combination of $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$.

By The Uniqueness Theorem, $\Phi$ **must** be a linear combination of these two solutions.

Thus these two particular solutions are a **Spanning Set** for the collection of all solutions to the homogeneous system.

The two particular solutions $e^{3t}\vec{v}$ and $e^{-20t}\vec{w}$ form a **Spanning Set** for the collection of all solutions to the homogeneous system.

What Made This Work?

$\vec{v}, \vec{w}$ is a linearly independent set of vectors which we know is true since they are associated with two distinct eigenvalues.

Moreover, the two solutions themselves are Linearly Independent Solutions. They form a **BASIS** for the set of all solutions to the homogeneous system of differential equations $\mathbf{X'} = A\,\mathbf{X}$.

Theorem: Let $\lambda$ and $\mu$ be distinct eigenvalues for a square matrix $A$ with corresponding eigenvectors $\vec{v}$ and $\vec{w}$. Then $e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}$ is a linearly independent set of solutions for $\mathbf{X'} = A\,\mathbf{X}$.

Theorem: Let $\lambda$ and $\mu$ be distinct eigenvalues for a square matrix $A$ with corresponding eigenvectors $\vec{v}$ and $\vec{w}$.

Then $\{e^{3t}\vec{v},\, e^{-20t}\vec{w}\}$ is a Linearly Independent set of solutions of $\mathbf{X'} = A\,\mathbf{X}$.

Proof: Suppose there are constants $C_1$ and $C_2$ such that

$$C_1 e^{\lambda t}\vec{v} + C_2 e^{\mu t}\vec{w} = \vec{0}$$

for all $t$ where $\vec{0}$ is the function identically equal to the zero vector for all $t$.

Evaluate this identity at $t = 0$:

$$C_1\vec{v} + C_2\vec{w} = 0$$

BUT $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors.
Hence it must be that $C_1 = 0$ and $C_2 = 0$

We used the fact that $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors to prove that

- $\{e^{3t}\vec{v},\ e^{-20t}\vec{w}\}$ is a spanning set for the solutions of $\mathbf{X'} = A\ \mathbf{X}$ and
- $\{e^{3t}\vec{v},\ e^{-20t}\vec{w}\}$ is a linearly independent set of solutions of $\mathbf{X'} = A\ \mathbf{X}$

The Linear Independence of $\{\vec{v}, \vec{w}\}$ followed from the fact that they were associated with distinct (unequal) eigenvalues.

**WE CAN DO THE SAME THING FOR ANY SUCH SYSTEM WHERE THE MATRIX HAS TWO DISTINCT EIGENVALUES**

# Another Linear Homogeneous System with Constant Coefficients

**X'** $= A$ **X** where $A$ is a $2 \times 2$ matrix.

$$x' = 2x + 1y$$
$$y' = -3x + 6y$$

$$A = \begin{bmatrix} 2 & 1 \\ -3 & 6 \end{bmatrix}$$

Characteristic Equation $(\det A - \lambda I) = 0$ is

$$\lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5) = 0$$

so eigenvalues are $\lambda = 3, \mu = 5$

and solution to the systems of first order differential equations is

$$C_1 e^{3t}\vec{v} + C_2 e^{5t}\vec{w}$$

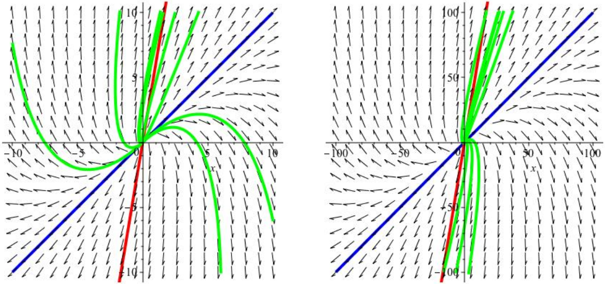
where $C_1, C_2$ are arbitrary constants and

$\vec{v}, \vec{w}$ are eigenvectors associated with $\lambda = 3$ and $\lambda = 5$, respectively.

We can write the general solution
$C_1 e^{3t}\vec{v} + C_2 e^{5t}\vec{w}$ as $e^{5t}(C_1 e^{-2t}\vec{v} + C_2\vec{w})$

If $C_2 = 0$, then solution does what as $t$ gets large?
Moves along the vector $\vec{v}$.

If $C_2 \neq 0$, then what does the solution do as $t$ gets large?
Approaches the vector $\vec{w}$.




$$x' = 2x + 1y,\ y' = -3x + 6y$$

$$C_1 e^{3t}\vec{v} + C_2 e^{5t}\vec{w} = e^{5t}(C_1 e^{-2t}\vec{v} + C_2\vec{w})$$