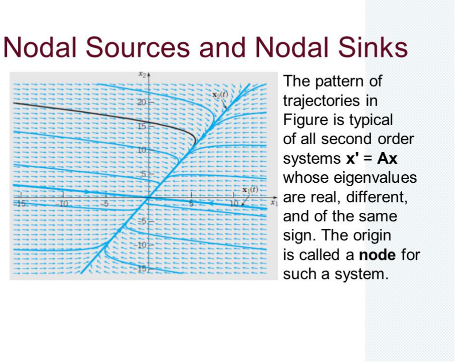
MATH 226: Differential Equations



Class 15: March 14, 2025


HAPPY
pi
π
3.14
DAY

Notes on Assignment 8

Assignment 9

In Handouts Folder:

PhasePlane Tutorial (also need PlotPhasePlane.m)

Announcements

For Next Time, Work Through

Complex Numbers

- Use the imaginary unit *i* to write complex numbers, and add, subtract, and multiply complex numbers.
- Find complex solutions of quadratic equations.
- Write the trigonometric forms of complex numbers.
- Find powers and *n*th roots of complex numbers.

Our Main Agenda

Solve **X' = A X** where A is an $n \times n$ matrix of constants and **X** is an n-dimensional vector of functions.

Results So Far

Theorem: The set of solutions is an n-dimensional vector space.

We can find some solutions of the form $e^{\lambda t}\vec{v}$ where λ is an eigenvalue of A and $\vec{v}$ is an associated eigenvector.

Distinct eigenvalues give rise to linearly independent solutions.

Outstanding Questions

How to handle complex eigenvalues.

How to find n linearly independent solutions to **X' = A X** when there are not enough of the form $e^{\lambda t}\vec{v}$. .

Comments on Some Homework Problems

Section 3.1 : Exercise 38

Show that $\lambda = 0$ is an eigenvalue for a matrix $\mathbf{A}$

if and only if

$\det(\mathbf{A}) = 0$.

Must Prove Both:

- If $\lambda = 0$ is an eigenvalue, then $\det(\mathbf{A}) = 0$, **AND**
- If $\det(\mathbf{A}) = 0$, then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

If $\lambda = 0$ is an eigenvalue, then det(**A**) =0

Proof:
If $\lambda = 0$ is an eigenvalue, then there is a nonzero vector $\vec{v}$ such that **A** $\vec{v} = \vec{0}$.
Hence **A** is not invertible so det(**A**) $= 0$.

If det(**A**) =0, then $\lambda = 0$ is an eigenvalue of **A**.

Proof: Note that **A** = **A** - **0** = **A** - 0 **I**.
If det(**A**) = 0, then det(**A** - 0 **I**) = 0
so 0 satisfies the characteristic equation for **A** and hence is an eigenvalue.

Comments on Some Homework Problems

Section 3.2 Exercise 30c

Use a computer to draw component plots of the initial value problem and the equilibrium solutions.

See *Problem 30 on Page 144.maple*

Current Goal:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

$$X' = A\,X$$

2×2 Case

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Characteristic Polynomial of A is det(A - λ I) $=$

$$\lambda^2 - (a+d)\lambda + ad - bc$$

$$\lambda^2 \text{ - Trace}(A)\ \lambda + \text{Det } A$$

Characteristic Equation: det(A - λ I) $= 0$

Eigenvalues Are Roots of Characteristic Polynomial

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

2 Real Unequal Roots

2 Complex Roots

1 Real Double Root

More About 2 Real Unequal Roots Case

$$\mathbf{x}' = A\,\mathbf{x}$$

The Origin (0,0) is an equilibrium and is called a **NODE**

More About 2 Real Unequal Roots Case $\mathbf{x}' = A\,\mathbf{x}$

The Origin (0,0) is an equilibrium and is called a **NODE**

$\lambda_1, \lambda_2 < 0$	Node is Asymptotically Stable **NODAL SINK**
$\lambda_1, \lambda_2 > 0$	Node is Unstable **NODAL SOURCE**
Opposite Sign	Node is Unstable **SADDLE POINT**

Nodal Sink	$\begin{bmatrix} -7 & 3 \\ 2 & -2 \end{bmatrix}$	$\lambda = -1, -8$
Nodal Source	$\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$	$\lambda = 3, 2$
Saddle Point	$\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$	$\lambda = 3, -1$

ZERO AS AN EIGENVALUE

Example: Richardson Arms Race Model
With Parallel Stable Lines

$$x' = -mx + ay + r$$
$$y' = bx - ny + s$$

Slope of **L** $= \frac{m}{a}$
Slope of **L'** $= \frac{b}{n}$

$$\text{Parallel if } \frac{m}{a} = \frac{b}{n}$$

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix}$$
$$mn = ab \Leftrightarrow mn - ab = 0 \Leftrightarrow det(A) = 0$$
Characteristic Equation: $\lambda^2 + (m+n)\lambda + (mn - ab) = 0$
$$\lambda^2 + (m+n)\lambda = 0$$
$$\lambda(\lambda + (m+n)) = 0 \Rightarrow \lambda = 0, \lambda = -(m+n)$$

ZERO AS AN EIGENVALUE EXAMPLE

$$m = 3, a = 6, n = 8, b = 4$$

$$A = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}$$

$$\det \begin{bmatrix} -3-\lambda & 6 \\ 4 & -8-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(-8-\lambda) - 24 = \lambda^2 + 11\lambda + 24 - 24$$

$$= \lambda^2 + 11\lambda = \lambda(\lambda + 11)$$

$$\lambda = 0, \lambda = -11$$

For $\lambda = -11$:

$$\begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4v_1 + 3v_2 = 0$$

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3w_1 + 6w_2 = 0$$

$$\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -11, \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\lambda = 0, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

General Solution To

$$x' = -3x + 6y$$

$$y' = 4x = 8y$$

is $\mathbf{x} = C_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-11t} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$x = 2C_1 - 3C_2 e^{-11t}$$

$$y = C_1 + 4C_2 e^{-11t}$$

Particular Solution: (x_0, y_0) at $t = 0$

$$x_0 = 2C_1 - 3C_2$$
$$y_0 = C_1 + 4C_2$$

$2C_1 - 3C_2 = x_0$	$8C_1 - 12C_2 = 4x_0$
$-2C_1 - 8C_2 = -2y_0$	$3C_1 + 12C_2 = 3y_0$
Add Equations	Add Equations
$-11C_2 = x_0 - 2y_0$	$11C_1 = 4x_0 + 3y_0$

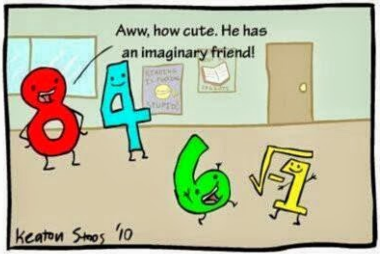
$$C_1 = \frac{4x_0 + 3y_0}{11}, C_2 = \frac{-x_0 + 2y_0}{11}$$

$$x = 2\left(\frac{4x_0 + 3y_0}{11}\right) - 3\left(\frac{-x_0 + 2y_0}{11}\right)e^{-11t}$$

$$y = \frac{4x_0 + 3y_0}{11} + 4\left(\frac{-x_0 + 2y_0}{11}\right)e^{-11t}$$

Next Time

Complex Eigenvalues

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