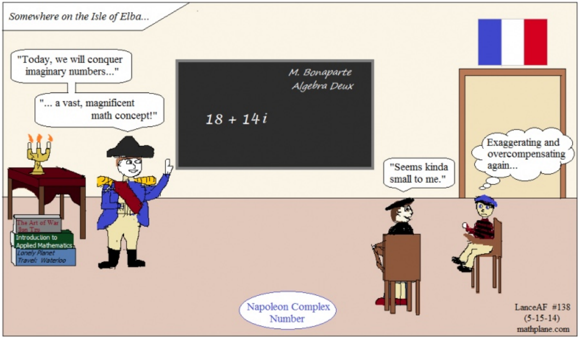
MATH 226:Differential Equations



Class 16: March 24, 2025

Notes on Assignment 10
Assignment 11

Our Main Agenda

Solve **X' = A X** where A is an $n \times n$ matrix of constants and **X** is an n-dimensional vector of functions.

Results So Far

Theorem: The set of solutions is an n-dimensional vector space.

We can find some solutions of the form $e^{\lambda t}\vec{v}$ where λ is an eigenvalue of A and $\vec{v}$ is an associated eigenvector.

Distinct eigenvalues give rise to linearly independent solutions.

Outstanding Questions

How to handle complex eigenvalues.

How to find n linearly independent solutions to **X' = A X** when there are not enough of the form $e^{\lambda t}\vec{v}$. .

Current Goal:

Continue Study of Linear Homogeneous Systems With Constant Coefficients

$X' = A\,X$

2×2 Case

Theorem: If λ and μ are distinct eigenvalues (real or complex) of a 2×2 matrix A having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x}' = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Characteristic Polynomial of A is det(A - λ I) =

$$\lambda^2 - (a+d)\lambda + ad - bc$$

λ^2 - Trace(A) λ + Det A

Characteristic Equation: det(A - λ I) = 0

Eigenvalues Are Roots of Characteristic Polynomial

$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

$$\lambda = \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Possibilities

2 Real Unequal Roots

2 Complex Roots

1 Real Double Root

More About 2 Real Unequal Roots Case

$$\mathbf{x}' = A\,\mathbf{x}$$

The Origin (0,0) is an equilibrium and is called a **NODE**

More About 2 Real Unequal Roots Case $\mathbf{x}' = A\,\mathbf{x}$

The Origin (0,0) is an equilibrium and is called a **NODE**

$\lambda_1, \lambda_2 < 0$	Node is Asymptotically Stable **NODAL SINK**
$\lambda_1, \lambda_2 > 0$	Node is Unstable **NODAL SOURCE**
Opposite Sign	Node is Unstable **SADDLE POINT**

Nodal Sink $\begin{bmatrix} -7 & 3 \\ 2 & -2 \end{bmatrix}$ $\lambda = -1, -8$

Nodal Source $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ $\lambda = 3, 2$

Saddle Point $\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $\lambda = 3, -1$

$\lambda = 0$ is an eigenvalue for a matrix **A**
if and only if
det(**A**) $=$0.

ZERO AS AN EIGENVALUE

Example: Richardson Arms Race Model
With Parallel Stable Lines

$$x' = -mx + ay + r$$
$$y' = bx - ny + s$$

Slope of **L** $= \frac{m}{a}$
Slope of **L'** $= \frac{b}{n}$

$$\text{Parallel if } \frac{m}{a} = \frac{b}{n}$$

$$A = \begin{bmatrix} -m & a \\ b & -n \end{bmatrix}$$
$$mn = ab \Leftrightarrow mn - ab = 0 \Leftrightarrow det(A) = 0$$
$$\text{Characteristic Equation: } \lambda^2 + (m+n)\lambda + (mn - ab) = 0$$
$$\lambda^2 + (m+n)\lambda = 0$$
$$\lambda(\lambda + (m+n)) = 0 \Rightarrow \lambda = 0, \lambda = -(m+n)$$

ZERO AS AN EIGENVALUE EXAMPLE

$$m = 3, a = 6, n = 8, b = 4$$

$$A = \begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix}$$

$$\det \begin{bmatrix} -3-\lambda & 6 \\ 4 & -8-\lambda \end{bmatrix}$$

$$= (-3-\lambda)(-8-\lambda) - 24 = \lambda^2 + 11\lambda + 24 - 24$$

$$= \lambda^2 + 11\lambda = \lambda(\lambda + 11)$$

$$\lambda = 0, \lambda = -11$$

For $\lambda = -11$:

$$\begin{bmatrix} 8 & 6 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4v_1 + 3v_2 = 0$$

$$\vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

For $\lambda = 0$:

$$\begin{bmatrix} -3 & 6 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3w_1 + 6w_2 = 0$$

$$\vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\lambda = -11, \vec{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

$$\lambda = 0, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

General Solution To

$$x' = -3x + 6y$$

$$y' = 4x = 8y$$

is $\mathbf{x} = C_1 e^{0t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-11t} \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$x = 2C_1 - 3C_2 e^{-11t}$$

$$y = C_1 + 4C_2 e^{-11t}$$

Particular Solution: (x_0, y_0) at $t = 0$

$$x_0 = 2C_1 - 3C_2$$

$$y_0 = C_1 + 4C_2$$

$$2C_1 - 3C_2 = x_0$$

$$-2C_1 - 8C_2 = -2y_0$$

Add Equations

$$-11C_2 = x_0 - 2y_0$$

$$8C_1 - 12C_2 = 4x_0$$

$$3C_1 + 12C_2 = 3y_0$$

Add Equations

$$11C_1 = 4x_0 + 3y_0$$

$$C_1 = \frac{4x_0 + 3y_0}{11}, C_2 = \frac{-x_0 + 2y_0}{11}$$

$$x = 2(\frac{4x_0 + 3y_0}{11}) - 3(\frac{-x_0 + 2y_0}{11})e^{-11t}$$

$$y = \frac{4x_0 + 3y_0}{11} + 4(\frac{-x_0 + 2y_0}{11})e^{-11t}$$

Consider the system of first order linear homogeneous differential equations

$$x'(t) = 2x(t) + py(t)$$
$$y'(t) = -1x(t) + 3y(t)$$

where p is any real number.

Then for any initial condition $x(0) = x_0, y(0) = y_0$, there is a unique solution of the system $x = f(t), y = g(t)$ satisfying the initial condition.

The values of $f(t)$ and $g(t)$ will be **real** numbers for all t.

Complex Eigenvalues

Begin with an example **X' = AX** where

$$A = \begin{pmatrix} 2 & p \\ -1 & 3 \end{pmatrix}$$

Here $det(A - \lambda I) = (2-\lambda)(3-\lambda) + p = \lambda^2 - 5\lambda + 6 + p$

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6+p)}}{2} = \frac{5 \pm \sqrt{1-4p}}{2}$$

Complex Eigenvalues

$$\lambda = \frac{5 \pm \sqrt{25 - 4(6+p)}}{2} = \frac{5 \pm \sqrt{1-4p}}{2}$$

Some Cases

1. p = 0: $\lambda = \frac{5 \pm 1}{2} = 3$ or 2 (source)
2. p = 1/4: $\lambda = \frac{5}{2}$ Double Root (Next Time)
3. **p = 5/2**: $\lambda = \frac{5 \pm \sqrt{1-10}}{2} = \frac{5 \pm \sqrt{-9}}{2} = \frac{5 \pm 3i}{2}$

   $\lambda = \frac{5+3i}{2}$ or $\lambda = \frac{5-3i}{2}$. (**Complex Conjugates**)

   $\lambda = \frac{5}{2} + \frac{3}{2}i$ or $\frac{5}{2} - \frac{3}{2}i$

For a quadratic polynomial, the quadratic formula shows we will have a conjugate pair of roots for $ax^2 + bx + c = 0$ when $b^2 - 4ac < 0$.

Some Basic Facts About Complex Numbers

A **complex number** z is an expression of the form $a + bi$ where a and b are real numbers and $i^2 = -1$.

a is called the real part of the complex number,

b is called the imaginary part.

Treat complex numbers as if they were real for the purposes of arithmetic except whenever you encounter ii, replace it with -1.

Arithmetic

Use Associative and Commutative Laws

$$z = a + bi, w = c + di$$

SUM: $z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$

PRODUCT

$$zw = (a+bi)(c+di) = ac+adi+bci+bdi^2 = (ac-bd)+(ad+bc)i$$

Powers of i

$$i^2 = -1, i^3 = i^2 i = -i, i^4 = i^2 i^2 = (-1)(-1) = 1$$

Thus
$+i = i^1 = i^5 = i^9 = i^{13} = i^{17} = ...$
$-1 = i^2 = i^6 = i^{10} = i^{14} = ...$
$-i = i^3 = i^7 = i^{11} = i^{15} = ...$
$+1 = i^4 = i^8 = i^{12} = i^{16} = ...$

In general, $i^k = i^{k+4}$.

Working with Conjugates

$\bar{z} = a - bi$

Then. $\overline{z+w} = \bar{z} + \bar{w}$ (Conjugate of sum is sum of conjugates)

$\overline{zw} = \bar{z}\bar{w}$. (Conjugate of product is product of conjugates)

Note $\overline{z^2} = \overline{zz} = \bar{z}\bar{z} = (\bar{z})^2$.

It follows that if

$$A\vec{v} = \lambda\vec{v}, \text{ then } A\bar{\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

$\overline{(A\vec{v})} = A\bar{\vec{v}}$ since A is real.

Thus

$$A\bar{\vec{v}} = (\overline{A\vec{v}}) = \overline{\lambda\vec{v}} = \bar{\lambda}\bar{\vec{v}}$$

If λ is an eigenvalue of A with eigenvector $\vec{v}$, then $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{\vec{v}}$

Theorem: If z is a root of a polynomial with real coefficients, then so is $\bar{z}$.

Example: Suppose z is a root of $x^7 - 4x^3 + \pi x - 7$

Then $z^7 - 4z^3 + \pi z - 7 = 0$

Hence $\overline{z^7 - 4z^3 + \pi z - 7} = \bar{0} = 0$

So $\overline{z^7} - \overline{4z^3} + \overline{\pi z} - \bar{7} = 0$

implying $(\bar{z})^7 - 4(\bar{z})^3 - \pi\bar{z} - 7 = 0$

How To Find Eigenvectors
Example:

$$A = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix}, \lambda = \frac{5}{2} \pm \frac{3}{2} i.$$

We want $\vec{v}$

$$A - \lambda I = \begin{pmatrix} 2 - \frac{5}{2} - \frac{3}{2} i & \frac{5}{2} \\ -1 & 3 - \frac{5}{2} - \frac{3}{2} i \end{pmatrix} using \lambda = \frac{5}{2} + \frac{3}{2} i$$

$$A - \lambda I = \begin{pmatrix} -\frac{1}{2} - \frac{3}{2} i & \frac{5}{2} \\ -1 & \frac{1}{2} - \frac{3}{2} i \end{pmatrix}$$

How To Find Eigenvectors

$$A - \lambda I = \begin{pmatrix} -\frac{1}{2}i - \frac{3}{2}i & \frac{5}{2} \\ -1 & \frac{1}{2} - \frac{3}{2}i \end{pmatrix}$$

First, Check that the determinant is 0:

$$\det (A - \lambda I) = (-\tfrac{1}{2}i - \tfrac{3}{2}i)(\tfrac{1}{2} - \tfrac{3}{2}i) - (-1)(\tfrac{5}{2})$$

$$= -1\tfrac{1}{4} + \tfrac{3}{4}i - \tfrac{3}{4}i - \tfrac{9}{4} + \tfrac{5}{2} = 0.$$

Second, to find a vector $\vec{v}$ with $(A - \lambda I)\vec{v} = \vec{0}$, use the second equation

$$-1v_1 + (\tfrac{1}{2} - \tfrac{3}{2}i)v_2 = 0$$

so $v_1 = \frac{(1-3i}{2} v_2$

Let $v_2 = 2$. Then $v_1 = 1 - 3i$ so $\vec{v} = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix}$

$$\vec{v} = \begin{pmatrix} 1-3i \\ 2 \end{pmatrix}$$

Finally, check that $A\vec{v} = (\frac{5}{2} + \frac{3}{2}i)\vec{v}$:

$$A\vec{v} = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \begin{pmatrix} 2-6i+5 \\ -1+3i+6 \end{pmatrix} = \begin{pmatrix} 7-6i \\ 5+3i \end{pmatrix}$$

and

$$\frac{5+3i}{2}\vec{v} = \frac{5+3i}{2}\begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{5+3i}{2}(1-3i) \\ \frac{5+3i}{2}(2) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(5-15i+3i+9) \\ 5+3i \end{pmatrix}$$

$$= \begin{pmatrix} 7-6i \\ 5+3i \end{pmatrix}$$