# MATH 226: Differential Equations



Euler's Identity

$$e^{i\pi} + 1 = 0$$

Class 17:    March 26, 2025

Procedure For Complex Eigenvalues
Some Power Series Representations

**Announcements**
Project 2: Blood – Brain Pharmacokinetic Model
Pages 201 – 202 of Our Text
Due: Monday, April 7

**Current Goal:**

**Continue Study of Linear Homogeneous Systems With Constant Real Coefficients X' = A X 2 × 2 Case**

<u>**Theorem**</u>: If $\lambda$ and $\mu$ are distinct eigenvalues (real or complex) of a $2 \times 2$ matrix $A$ having corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then every solution of $\mathbf{x'} = A\,\mathbf{x}$ is a linear combination of $e^{\lambda t}\vec{v}$ and $e^{\mu t}\vec{w}$.

So Far:

- ▶ A has unequal real roots (Sources, Sinks, Saddle Points)
- ▶ Complex Eigenvalues and Eigenvectors

Consider the system of first order linear homogeneous differential equations

$$x'(t) = 2x(t) + py(t)$$
$$y'(t) = -1x(t) + 3y(t)$$

where $p$ is any real number.

Then for any initial condition $x(0) = x_0, y(0) = y_0$, there is a unique solution of the system $x = f(t), y = g(t)$ satisfying the initial condition.

The values of $f(t)$ and $g(t)$ will be **real** numbers for all $t$.

The pair $(f(t), g(t))$ parametrize a curve in the plane called the **trajectory** or **orbit** of the system.

## Apply To System of Differential Equations

$$X' = AX \text{ with } A = \begin{pmatrix} 2 & \frac{5}{2} \\ -1 & 3 \end{pmatrix}$$

We have

$$\lambda = \frac{5+3i}{2} \quad \text{so} \quad \mu = \frac{5-3i}{2}$$

$$\vec{v} = \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} 1 + 3i \\ 2 \end{pmatrix}$$

Solutions of Differential Equations Should be

$$e^{(\frac{5+3i}{2})t} \begin{pmatrix} 1 - 3i \\ 2 \end{pmatrix} \text{ and } e^{(\frac{5-3i}{2})t} \begin{pmatrix} 1 + 3i \\ 2 \end{pmatrix}$$

How Can We Make Sense of

$$e^{(\frac{5+3i}{2})t} = e^{(\frac{5}{2}t + \frac{3i}{2}t)}?$$

How Can We Make Sense of

$$e^{(\frac{5+3i}{2})t} = e^{(\frac{5}{2}t + \frac{3i}{2}t)}?$$

By Rules of Exponents $e^{a+b} = e^a e^b$, We Should Have

$$e^{(\frac{5}{2}t + \frac{3i}{2}t)} = e^{\frac{5}{2}t} e^{\frac{3}{2}it} = e^{\frac{5}{2}t} e^{(\frac{3t}{2})i}$$

# **Euler's Formula:**

$$e^{bi} = \cos b + i \sin b$$

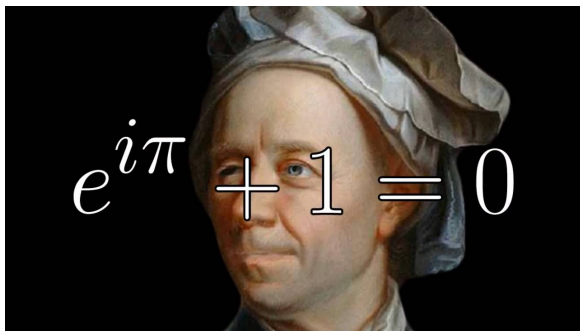so $e^{\frac{3}{2}it} = \cos \frac{3}{2}t + i \sin \frac{3}{2}t$

# Euler's Formula:

$$e^{bi} = \cos b + i \sin b$$

Note: If $b = \pi$, then

$$e^{\pi i} = \cos \pi + i \sin \pi = -1 + i0 = -1$$

$$\boxed{e^{\pi i} + 1 = 0}$$

Thus

$$e^{\frac{5}{2}t+\frac{3}{2}it}\begin{pmatrix} 1-3i \\ 2 \end{pmatrix} = e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\begin{pmatrix} 1-3i \\ 2 \end{pmatrix}$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} -3i \\ 0 \end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\cos\frac{3}{2}t + i\sin\frac{3}{2}t\right]\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix} + i\begin{pmatrix} -3 \\ 0 \end{pmatrix}\right]$$

$$= e^{\frac{5}{2}t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\cos\frac{3}{2}t - \begin{pmatrix} -3 \\ 0 \end{pmatrix}\sin\frac{3}{2}t\right] + ie^{\frac{5}{2}t}\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\sin\frac{3}{2}t + \begin{pmatrix} -3 \\ 0 \end{pmatrix}\cos\frac{3}{2}t\right]$$

$$= e^{\frac{5}{2}t} \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos \frac{3}{2}t - \begin{pmatrix} -3 \\ 0 \end{pmatrix} \sin \frac{3}{2}t \right] + i e^{\frac{5}{2}t} \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin \frac{3}{2}t + \begin{pmatrix} -3 \\ 0 \end{pmatrix} \cos \frac{3}{2}t \right]$$

**REAL PART:** $e^{\frac{5}{2}t} \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cos \frac{3}{2}t + \begin{pmatrix} 3 \\ 0 \end{pmatrix} \sin \frac{3}{2}t \right]$

**IMAGINARY PART:** $e^{\frac{5}{2}t} \left[ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \sin \frac{3}{2}t + \begin{pmatrix} -3 \\ 0 \end{pmatrix} \cos \frac{3}{2}t \right]$

# EACH PART SEPARATELY IS A SOLUTION

**Theorem**: Suppose $\overrightarrow{\phi(t)} = \overrightarrow{f(t)} + i\overrightarrow{g(t)}$ is a solution to $X' = AX$. Then $\overrightarrow{f(t)}$ and $\overrightarrow{g(t)}$ separately are solutions.

<u>Proof</u>: $\overrightarrow{\phi'(t)} = A\overrightarrow{\phi(t)}$ since $\overrightarrow{\phi(t)}$ is a solution.

Write $\overrightarrow{\phi'} = A\overrightarrow{\phi}$ for short. Thus

$$\overrightarrow{\phi'} - A\overrightarrow{\phi} = \overrightarrow{0}$$

$$[\overrightarrow{f'} + i\overrightarrow{g'}] - A[\overrightarrow{f} + i\overrightarrow{g}] = \overrightarrow{0}$$
$$[\overrightarrow{f'} - A\overrightarrow{f}] + i[\overrightarrow{g'} - A\overrightarrow{g}] = \overrightarrow{0}$$

But a complex number or vector of complex numbers is zero if and only both real and imaginary parts are zero.

Hence

$$\overrightarrow{f'} - A\overrightarrow{f} = \overrightarrow{0} \text{ so } \overrightarrow{f'} = A\overrightarrow{f}$$

and

$$\overrightarrow{g'} - A\overrightarrow{g} = \overrightarrow{0} \text{ so } \overrightarrow{g'} = A\overrightarrow{g}$$

Thus both $\overrightarrow{f}$ and $\overrightarrow{g}$ are solutions.

## Procedure for Finding The General Solution of $\mathbf{x' = Ax}$
## Where $A$ has Complex Eigenvalues

1. Identify the complex conjugate eigenvalues $\lambda = \mu \pm iv$.

2. Determine an eigenvector $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corresponding to $\lambda = \mu + iv$ by solving $(A - \lambda \mathbf{I})\mathbf{v} = \mathbf{0}$

3. Express the eigenvector $\mathbf{v}$ in the form $\mathbf{v} = \mathbf{p} + \mathbf{r}i$.

4. Write the solution $\mathbf{x}$ corresponding to $\mathbf{v}$ and separate it into real and imaginary parts:

$$\mathbf{x}(t) = \mathbf{u}(t) + i\,\mathbf{w}(t) \text{ where}$$

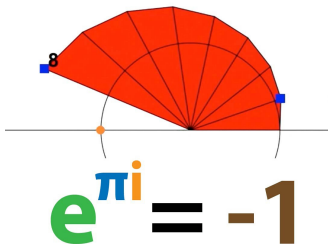$$\mathbf{u}(t) = e^{\mu t}(\mathbf{p}\cos vt - \mathbf{r}\sin vt) \text{ and}$$

$$\mathbf{w}(t) = e^{\mu t}(\mathbf{r}\cos vt + \mathbf{p}\sin vt)$$

Then $\mathbf{u}$ and $\mathbf{w}$ form a linearly independent set of solutions for $\mathbf{x' = Ax.}$

5. The general solution of $\mathbf{x' = Ax}$ is $\mathbf{x}(t) = C_1\,\mathbf{u}(t) + C_2\,\mathbf{w}(t)$ where $C_1$ and $C_2$ are arbitrary constants.

# Euler's Identity

$$e^{\pi i} = -1$$

# Where Did Euler Get The Idea that $e^{i\theta} = \cos\theta + i\sin\theta$?

$$e^x \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;+\; \ldots$$

$$\;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$x \in \mathbb{R}$

> SO:
> $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$
> $e^{(17x)} = \sum_{n=0}^{\infty} \frac{(17x)^n}{n!} = \sum_{n=0}^{\infty} \frac{17^n x^n}{n!}$

$$\cos x \;=\; 1 \;-\; \frac{x^2}{2!} \;+\; \frac{x^4}{4!} \;-\; \frac{x^6}{6!} \;+\; \frac{x^8}{8!} \;-\; \ldots$$

$$\;=\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$x \in \mathbb{R}$

> NOTE $y = \cos x$ IS AN <u>EVEN</u> FUNCTION (I.E., $\cos(-x) = +\cos(x)$) AND THE TAYLOR SERIS OF $y = \cos x$ HAS ONLY <u>EVEN</u> POWERS.

$$\sin x \;=\; x \;-\; \frac{x^3}{3!} \;+\; \frac{x^5}{5!} \;-\; \frac{x^7}{7!} \;+\; \frac{x^9}{9!} \;-\; \ldots$$

$$\;=\; \sum_{n=1}^{\infty} (-1)^{(n-1)} \frac{x^{2n-1}}{(2n-1)!} \;\overset{\text{or}}{\equiv}\; \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$x \in \mathbb{R}$

> NOTE $y = \sin x$ IS AN <u>ODD</u> FUNCTION (I.E., $\sin(-x) = -\sin(x)$) AND THE TAYLOR SERIS OF $y = \sin x$ HAS ONLY <u>ODD</u> POWERS.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \frac{x^8}{8!} + \frac{x^9}{9!} + \frac{x^{10}}{10!} + \frac{x^{11}}{11!} + \frac{x^{12}}{12!} + ...$$

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \frac{(ix)^9}{9!} + \frac{(ix)^{10}}{10!} + \frac{(ix)^{11}}{11!} + \frac{(ix)^{12}}{12!} + ...$$

$$e^{ix} = 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \frac{i^6 x^6}{6!} + \frac{i^7 x^7}{7!} + \frac{i^8 x^8}{8!} + \frac{i^9 x^9}{9!} + \frac{i^{10} x^{10}}{10!} + \frac{i^{11} x^{11}}{11!} + \frac{i^{12} x^{12}}{12!} + ...$$

$$e^{ix} = 1 + ix + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} + \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} + \frac{-x^{10}}{10!} + \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + ...$$

$$e^{ix} =$$
$$1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{-x^6}{6!} + \frac{-ix^7}{7!} + \frac{x^8}{8!} + \frac{ix^9}{9!} - \frac{-x^{10}}{10!} - \frac{-ix^{11}}{11!} + \frac{x^{12}}{12!} + ...$$

$$e^{ix} =$$
$$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + ...$$

$e^{ix} =$

$1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \frac{x^8}{8!} + i\frac{x^9}{9!} - \frac{x^{10}}{10!} - i\frac{x^{11}}{11!} + \frac{x^{12}}{12!} + ...$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + ...$
$+ ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + i\frac{x^9}{9!} - i\frac{x^{11}}{11!} + ...$

$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \frac{x^{12}}{12!} + ...$
$+ i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ...)$

$= \cos x + i \sin x$

# Repeated Eigenvalues

$$A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$$

$$det(A - \lambda I) = det \begin{pmatrix} 3 - \lambda & -4 \\ 1 & -1 - \lambda \end{pmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$$

Eigenvector: $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ so $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

One Solution of $\mathbf{X}' = A\mathbf{X}$ is $e^t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

**How do we find a second solution?**