MATH 226: Differential Equations

Class 19: March 31, 2025

Notes on Assignment 12

Assignment 13

Team Members for Project 2

Political Movement Model in *Maple* (in Handouts Folder)

Political Movement Model in MATLAB

Announcements

- Second Project Due Monday, April 7
- Exam 2 on Wednesday, April 16

Mathematician of the Week: **Mary Lee Wheat Gray**

April 4, 1939 –

Mary Gray is an American mathematician, statistician, and lawyer. She has written on mathematics, education, computer science, statistics and academic freedom.

Systems of First Order Linear Differential Equations

Why Not Study Second Order Equations?

Damped Harmonic Oscillator	Swinging Pendulum
$mw''(t) + bw' + kw = 0$	$\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$

Let $x = w$ and $y = w'$.

Then $x' = w' = y$ and $y' = w''$

so $mw''(t) + bw' + kw = 0$ becomes $my' + by + kx = 0$

Thus we have the system

$$x' = y$$

$$y' = -\frac{k}{m}x - \frac{b}{m}y$$

Let $x = \theta$ and $y = \theta'$. Then $\theta''(t) + \frac{g}{L}\sin\theta(t) = 0$ becomes system $x' = y, y' + \frac{g}{l}\sin x = 0$.

Systems of First Order Linear Differential Equations

$$
\begin{aligned}
x' &= (\sin t)x + \left(\frac{1}{t}\right) y + 9z + 2t^3 \\
y' &= (t^2)x - (\cos 3t)y + (e^{-3t})z + \sec t \\
z' &= (\log t)x - 2020y + (\tan t)z + e^{4t^2}
\end{aligned}
$$

$$
\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sin t & \frac{1}{t} & 9 \\ t^2 & -\cos 3t & e^{-3t} \\ \log t & -2020 & \tan t \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 2t^3 \\ \sec t \\ e^{4t^2} \end{pmatrix}
$$

$$\mathbf{X'} = P(t)\,\mathbf{X} + \mathbf{g}(\mathrm{t})$$

Homogeneous: $\mathbf{X'} = P(t)\,\mathbf{X}$

Major Theorems On Systems of First Order Linear Differential Equations

Basic Existence and Uniqueness Result

THEOREM 6.2.1

(Existence and Uniqueness for First Order Linear Systems). If $\mathbf{P}(t)$ and $\mathbf{g}(t)$ are continuous on an open interval $I = (\alpha, \beta)$, then there exists a unique solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{2}$$

where t_0 is any point in I, and $\mathbf{x}_0$ is any constant vector with n components. Moreover the solution exists throughout the interval I.

Linear Combinations of Solutions of Homogeneous Systems Are Solutions

THEOREM 6.2.2

(Principle of Superposition). If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are solutions of the homogeneous linear system

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{5}$$

on the interval $I = (\alpha, \beta)$, then the linear combination

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$$

is also a solution of Eq. (5) on I.

Proof

Let $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$. The result follows from the linear operations of matrix multiplication and differentiation:

$$\begin{aligned}
\mathbf{P}(t)\mathbf{x} &= \mathbf{P}(t)[c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k] \\
&= c_1\mathbf{P}(t)\mathbf{x}_1 + \cdots + c_k\mathbf{P}(t)\mathbf{x}_k \\
&= c_1\mathbf{x}_1' + \cdots + c_k\mathbf{x}_k' = \mathbf{x}'.
\end{aligned}$$

Definition of Linear Independence

DEFINITION 6.2.3

The n vector functions $\mathbf{x}_1, \dots, \mathbf{x}_n$ are said to be **linearly independent on an interval** I if the only constants $c_1, c_2, \dots, c_n$ such that

$$c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t) = \mathbf{0} \tag{6}$$

for all $t \in I$ are $c_1 = c_2 = \cdots = c_n = 0$. If there exist constants $c_1, c_2, \dots, c_n$, *not all zero*, such that Eq. (6) is true for all $t \in I$, the vector functions are said to be **linearly dependent** on I.

Jozef Maria Hoene Wronski
Józef Maria Hoene-Wroński
1776 –1853

Wronskians and the Struggle for Linear Independence

DEFINITION 6.2.4 Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be n solutions of the homogeneous linear system of differential equations $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ and let $\mathbf{X}(t)$ be the $n \times n$ matrix whose jth column is $\mathbf{x}_j(t)$, $j = 1, \ldots, n$,

$$\mathbf{X}(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix}. \tag{12}$$

The **Wronskian** $W = W[\mathbf{x}_1, \ldots, \mathbf{x}_n]$ of the n solutions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is defined by

$$W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det \mathbf{X}(t). \tag{13}$$

THEOREM 6.2.5

Let $\mathbf{x}_1, \dots, \mathbf{x}_n$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an interval $I = (\alpha, \beta)$ in which $\mathbf{P}(t)$ is continuous.

(i) If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I, then $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) \neq 0$ at every point in I,

(ii) If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent on I, then $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) = 0$ at every point in I.

Proof

Assume first that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I. We then want to show that $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) \neq 0$ throughout I. To do this, we assume the contrary, that is, there is a point $t_0 \in I$ such that $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t_0) = 0$. This means that the column vectors $\{\mathbf{x}_1(t_0), \dots, \mathbf{x}_n(t_0)\}$ are linearly dependent (Theorem A.3.6) so that there exist constants $\hat{c}_1, \dots, \hat{c}_n$, not all zero, such that $\hat{c}_1\mathbf{x}_1(t_0) + \cdots + \hat{c}_n\mathbf{x}_n(t_0) = \mathbf{0}$. Then Theorem 6.2.2 implies that $\boldsymbol{\phi}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t)$ is a solution of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ that satisfies the initial condition $\mathbf{x}(t_0) = \mathbf{0}$. The zero solution also satisfies the same initial value problem. The uniqueness part of Theorem 6.2.1 therefore implies that $\boldsymbol{\phi}$ is the zero solution, that is, $\boldsymbol{\phi}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t) = \mathbf{0}$ for every $t \in (\alpha, \beta)$, contradicting our original assumption that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly independent on I. This proves (i).

To prove (ii), assume that $\mathbf{x}_1, \dots, \mathbf{x}_n$ are linearly dependent on I. Then there exist constants $\alpha_1, \dots, \alpha_n$, not all zero, such that $\alpha_1\mathbf{x}_1(t) + \cdots + \alpha_n\mathbf{x}_n(t) = \mathbf{0}$ for every $t \in I$. Consequently, for each $t \in I$, the vectors $\mathbf{x}_1(t), \dots, \mathbf{x}_n(t)$ are linearly dependent. Thus $W[\mathbf{x}_1, \dots, \mathbf{x}_n](t) = 0$ at every point in I (Theorem A.3.6).

Dimension of Solution Space of x' = P(t) x

THEOREM 6.2.6

Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be solutions of

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x} \tag{14}$$

on the interval $\alpha < t < \beta$ such that, for some point $t_0 \in (\alpha, \beta)$, the Wronskian is nonzero, $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$. Then each solution $\mathbf{x} = \boldsymbol{\phi}(t)$ of Eq. (14) can be expressed as a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_n$,

$$\boldsymbol{\phi}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t), \tag{15}$$

where the constants $\hat{c}_1, \ldots, \hat{c}_n$ are uniquely determined.

Proof

Let $\boldsymbol{\phi}(t)$ be a given solution of Eq. (14). If we set $\mathbf{x}_0 = \boldsymbol{\phi}(t_0)$, then the vector function $\boldsymbol{\phi}$ is a solution of the initial value problem

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{16}$$

By the principle of superposition, the linear combination $\boldsymbol{\psi}(t) = c_1\mathbf{x}_1(t) + \cdots + c_n\mathbf{x}_n(t)$ is also a solution of (14) for any choice of constants $c_1, \ldots, c_n$. The requirement $\boldsymbol{\psi}(t_0) = \mathbf{x}_0$ leads to the linear algebraic system

$$\mathbf{X}(t_0)\mathbf{c} = \mathbf{x}_0, \tag{17}$$

where $\mathbf{X}(t)$ is defined by Eq. (12). Since $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t_0) \neq 0$, the linear algebraic system (17) has a unique solution (see Theorem A.3.7) that we denote by $\hat{c}_1, \ldots, \hat{c}_n$. Thus the particular member $\hat{\boldsymbol{\psi}}(t) = \hat{c}_1\mathbf{x}_1(t) + \cdots + \hat{c}_n\mathbf{x}_n(t)$ of the n-parameter family represented by $\boldsymbol{\psi}(t)$ also satisfies the initial value problem (16). By the uniqueness part of Theorem 6.2.1, it follows that $\boldsymbol{\phi} = \hat{\boldsymbol{\psi}} = \hat{c}_1\mathbf{x}_1 + \cdots + \hat{c}_n\mathbf{x}_n$. Since $\boldsymbol{\phi}$ is arbitrary, the result holds (with different constants, of course) for every solution of Eq. (14).

THEOREM 6.2.7

Let

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix};$$

further let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ that satisfy the initial conditions

$$\mathbf{x}_1(t_0) = \mathbf{e}_1, \quad \ldots, \quad \mathbf{x}_n(t_0) = \mathbf{e}_n,$$

respectively, where t_0 is any point in $\alpha < t < \beta$. Then $\mathbf{x}_1, \ldots, \mathbf{x}_n$ form a fundamental set of solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$.

Homogenous Linear Systems With Constant Coefficients

X' $= P(\mathrm{t})$ **X** where $P(\mathrm{t})$ is a matrix of CONSTANTS

X' $= A$ **X** where A is an $n \times n$ matrix of CONSTANTS

$$
\begin{aligned}
x' &= 5x + 29y - 4z - 1w \\
y' &= 12x + 21y - 19z + 66w \\
z' &= -8x + 15y + 7z - 2w \\
w' &= 4x + 9y + 20z + 20w
\end{aligned}
$$

Linear Systems with Constant Coefficients

Simplest Case

THEOREM 6.3.1

Let $(\lambda_1, \mathbf{v}_1), \dots, (\lambda_n, \mathbf{v}_n)$ be eigenpairs for the real, $n \times n$ constant matrix A. Assume that the eigenvalues $\lambda_1, \dots, \lambda_n$ are real and that the corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent. Then

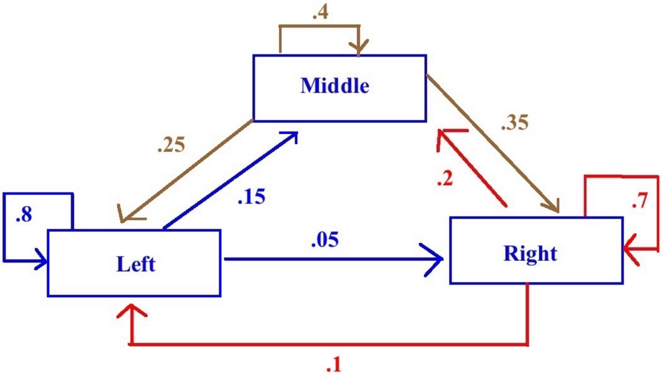
$$\left\{e^{\lambda_1 t}\mathbf{v}_1, \dots, e^{\lambda_n t}\mathbf{v}_n\right\} \tag{6}$$

is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval $(-\infty, \infty)$. The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is therefore given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n, \tag{7}$$

where $c_1, \dots, c_n$ are arbitrary constants.

A Differential Equations Model of Political Movement



$$L' = -.2L + .25M + .1R$$
$$M' = .15L - .6M + .2R$$
$$R' = .05L + .35M - .3R$$

Consider a system of first order linear homogeneous differential equations with constant coefficients

$$\mathbf{X'} = A\ \mathbf{X}$$

where A is $n \times n$ matrix of constants and **X** is $n \times 1$ vector of functions of t.

<u>Theorem 1</u> If λ is an eigenvalue of A with corresponding eigenvector v, then $e^{\lambda t} v$ is a solution of X' = AX.

<u>Proof</u>: If $\mathbf{X} = e^{\lambda t}\vec{v}$, then

$$\begin{aligned} \mathbf{X}' &= \lambda e^{\lambda t}\vec{v} \\ &= e^{\lambda t}\lambda\vec{v} \\ &= e^{\lambda t}\ A\ \vec{v} \\ &= A\ e^{\lambda t}\vec{v} \\ &= A\ \mathbf{X} \end{aligned}$$

Theorem 2 If λ and μ are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$ (that is, A $\vec{v} = \lambda$ $\vec{v}$ and A$\vec{w} = \mu\vec{w}$) then

1. $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}\}$ is a linearly independent set of solutions of **X'** = A**X**

Proof of 1: Suppose C1 and C2 are constants such that
(*) C1 $\vec{v}$ + C2 $\vec{w} = \vec{0}$.
Multiply (*) by A to obtain (**) C1 λ $\vec{v}$ + C2 μ $\vec{w} = \vec{0}$
Multiply (*) by μ to obtain (***) C1 μ $\vec{v}$ + C2 μ $\vec{w} = \vec{0}$
Subtract (***) from (**)to obtain C1(λ - μ) $\vec{v} = \vec{0}$
But λ - $\mu \neq 0$ and $\vec{v} \neq 0$; Hence C1 = 0
which implies C2 $\vec{w}$ =$\vec{0}$ and that implies C2 = 0.

Theorem 2 If λ and μ are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$ and $\vec{w}$, then

1. $\{\vec{v}, \vec{w}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}\}$ is a linearly independent set of solutions of **X'** = A**X**

Proof of 2: Suppose C1 and C2 are constants such that

$$C1\ e^{\lambda t}\vec{v} + C2\ e^{\mu t}\vec{w} = \vec{0}.$$

Evaluate both sides at $t = 0$:

$$C1\ e^{\lambda 0}\vec{v} + C2\ e^{\mu 0}\vec{w} = \vec{0}$$

$$C1\ e^{0}\vec{v} + C2\ e^{0}\vec{w} = \vec{0}$$

$$C1\ \vec{v} + C2\ \vec{w} = \vec{0}$$

which implies C1 and C2 are both 0.

A Generalization of Theorem 2

Theorem 3 If λ , μ and α are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v}$, $\vec{w}$ and $\vec{u}$
(that is, A $\vec{v} = \lambda\ \vec{v}$, A$\vec{w} = \mu\vec{w}$, A$\vec{u} = \alpha\vec{u}$)
then

1. $\{\vec{v}, \vec{w}, \vec{u}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda t}\vec{v}, e^{\mu t}\vec{w}, e^{\alpha t}\vec{u}\}$ is a linearly independent set of solutions of **X'** $=$ A**X**

A Even Bigger Generalization of Theorem 2

Theorem 4 If $\lambda_1, \lambda_2, ..., \lambda_k$, are **distinct** eigenvalues of A with corresponding eigenvectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_k}$
(that is, A $\vec{v_i} = \lambda_i$ for each $i = 1, 2, 3, ..., k$
then

1. $\{\vec{v_1}, \vec{v_2}, ..., \vec{v_k}\}$ is a linearly independent set of vectors
2. $\{e^{\lambda_1 t}\vec{v_1}, e^{\lambda_2 t}\vec{v_2}, ..., e^{\lambda_k t}\vec{v_k}\}$ is a linearly independent set of solutions of **X'** = A**X**