MATH 226: Differential Equations

Class 20: April 2, 2025

Political Movement Model in MATLAB

A 3 x 3 Example in *Maple* (Handouts Folder)

Announcements

Project Two Due Monday, April 7
Exam 2 on Wednesday, April 16

Theorem from Last Time:

Suppose λ, μ, ρ are distinct eigenvalues of the $n \times n$ matrix A with corresponding eigenvectors $\mathbf{v}, \mathbf{w}, \mathbf{u}$, respectively; that is

$$A\mathbf{v} = \lambda\mathbf{v}$$
$$A\mathbf{w} = \mu\mathbf{w}$$
$$A\mathbf{u} = \rho\mathbf{u}.$$

Then the set $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ is linearly independent.

A Major Generalization:

Let $\lambda_1, \lambda_2, ..., \lambda_m$ be m distinct eigenvalues of a square matrix A with corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}$, respectively; that is,

$$A\mathbf{v_i} = \lambda_i\mathbf{v_i} \text{ for } i = 1, 2, 3, ..., m.$$

Then the set $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}\}$ is linearly independent.

The four steps of math induction:

1. Show $P(1)$ is true

   Let $n = 1$ and work it out.

2. Assume $P(k)$ is true

   Stick a k in for all the n's and say it's true.

3. Show $P(k) \rightarrow P(k+1)$

   * In math, the arrow $\rightarrow$ means "implies" or "leads to."

   USE $P(k)$ to show that $P(k+1)$ is true.

   Very important!

4. End the proof

   Write "Thus, $P(n)$ is true." $\blacksquare$

   This is the modern way to end a proof.

Theorem: Let $\lambda_1, \lambda_2, ..., \lambda_m$ be m distinct eigenvalues of a square matrix A with corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}$, respectively; that is, $A\mathbf{v_i} = \lambda_i \mathbf{v_i}$ for $i = 1, 2, 3, ..., m$. Then the set $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}\}$ is linearly independent.

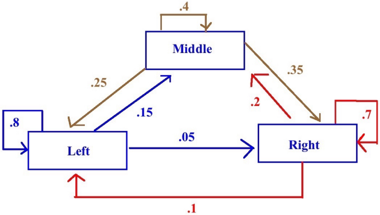
Consequently, the functions $e^{\lambda_1 t}\mathbf{v_1}, e^{\lambda_2 t}\mathbf{v_2}, ...e^{\lambda_m t}\mathbf{v_m}$ form a linearly independent set of solutions to the system $\mathbf{x}' = A\mathbf{x}$.

Here is a different generalization:

Suppose λ and μ are distinct eigenvalues of a square matrix A. The eigenvalue λ has associated with it a set of 3 linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ while the eigenvalue μ has an associated set of 2 eigenvectors $\mathbf{w_1}, \mathbf{w_2}$.
Then the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{w_1}, \mathbf{w_2}\}$ is linearly independent.

Theorem: Suppose λ and μ are distinct eigenvalues of a square matrix A. The eigenvalue λ has associated with it a set of 3 linearly independent eigenvectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ while the eigenvalue μ has an associated set of 2 eigenvectors $\mathbf{w_1}, \mathbf{w_2}$.
Then the set $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{w_1}, \mathbf{w_2}\}$ is linearly independent.
Proof: Suppose C_1, C_2, C_3, $C4$, C_5 are constants such that
$(*) C_1\mathbf{v_1} + C_2\mathbf{v_2} + C_3\mathbf{v_3} + C_4\mathbf{w_1} + C_5\mathbf{w_2} = 0$
Multiply (*) by A to obtain
$C_1A\mathbf{v_1} + C_2A\mathbf{v_2} + C_3A\mathbf{v_3} + C_4A\mathbf{w_1} + AC_5\mathbf{w_2} = A0$ or
$(**) C_1\lambda\mathbf{v_1} + C_2\lambda\mathbf{v_2} + C_3\lambda\mathbf{v_3} + C_4\mu\mathbf{w_1} + C_5\mu\mathbf{w_2} = 0$
Also multiply (*) by λ to obtain:
(***) $C_1\lambda\mathbf{v_1} + C_2\lambda\mathbf{v_2} + C_3\lambda\mathbf{v_3} + C_4\lambda\mathbf{w_1} + C_5\lambda\mathbf{w_2} = 0$
Subtract equation (***) from equation (**):
$C_4(\mu - \lambda)\mathbf{w_1} + C_5(\mu - \lambda)\mathbf{w_2} = 0$
But $\{\mathbf{w_1}, \mathbf{w_2}\}$ is a linearly independent set so $C_4 = 0, C_5 =$ since $\lambda \neq \mu$.
Substituting back into (*), we have $(*) C_1\mathbf{v_1} + C_2\mathbf{v_2} + C_3\mathbf{v_3} + = 0$
Linear Independence of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ now implies $C_1 = C_2 = C_3 0$ as well.

Political Movement Model



$$L' = -.2L + .25M + .1R = -\frac{1}{5}L + \frac{1}{4}M + \frac{1}{10}R$$

$$M' = .15L - .6M + .2R = \frac{3}{20}L - \frac{3}{5}M + \frac{1}{5}R$$

$$R' = .05L + .35M - .3R = \frac{1}{20}L + \frac{7}{20}M - \frac{3}{10}R$$

$$L' = -\frac{1}{5}L + \frac{1}{4}M + \frac{1}{10}R$$
$$M' = \frac{3}{20}L - \frac{3}{5}M + \frac{1}{5}R$$
$$R' = \frac{1}{20}L + \frac{7}{20}M - \frac{3}{10}R$$

$$\begin{pmatrix} L \\ M \\ R \end{pmatrix}' = \begin{pmatrix} -\frac{1}{5} & \frac{1}{4} & \frac{1}{10} \\ \frac{3}{20} & -\frac{3}{5} & \frac{1}{5} \\ \frac{1}{20} & \frac{7}{20} & -\frac{3}{10} \end{pmatrix} \begin{pmatrix} L \\ M \\ R \end{pmatrix}$$
$$\begin{pmatrix} L \\ M \\ R \end{pmatrix}' = A \begin{pmatrix} L \\ M \\ R \end{pmatrix}$$

Characteristic Polynomial $\det(A-\lambda I) = \lambda^3 + \frac{11}{10}\lambda^2 + \frac{99}{400}\lambda$

$$= \lambda\left(\lambda^2 + \frac{11}{10}\lambda + \frac{99}{400}\right)$$

Eigenvalues:

$$\lambda = 0$$

$$\lambda = \frac{-\frac{11}{20} \pm \sqrt{\frac{121}{100} - \frac{99}{100}}}{2} = \frac{-11 \pm \sqrt{22}}{20} = \begin{cases} \frac{-11+\sqrt{22}}{20} \approx -.315 \\ \frac{-11-\sqrt{22}}{20} \approx -.784 \end{cases}$$

Eigenvalue	Eigenvector
$\lambda = 0$	$\mathbf{v}$
$\lambda = -.315$	$\mathbf{w}$
$\lambda = -.784$	$\mathbf{u}$

General Solution: $C_1 e^{0t}\mathbf{v} + C_2 e^{-.315t}\mathbf{w} + C_2 e^{-.784t}\mathbf{u}$

General Solution: $\mathbf{X}(t) = C_1 e^{0t}\mathbf{v} + C_2 e^{-.315t}\mathbf{w} + C_2 e^{-.784t}\mathbf{u}$

$$\mathbf{X}(t) = C_1\mathbf{v} + C_2 e^{-.315t}\mathbf{w} + C_2 e^{-.784t}\mathbf{u}$$

As $t \to \infty, \mathbf{X}(t) \to C_1\mathbf{v}$
Important to find $\mathbf{v}$

$\mathbf{v}$ is scalar multiple of $\begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$

Entries in $\mathbf{X}(t)$ must add to 1:
$4c + 2c + 3c = 1$ implies $9c = 1; c = 1/9$

Thus $\mathbf{X}(t) \to \begin{pmatrix} 4/9 \\ 2/9 \\ 3/9 \end{pmatrix}$

Another 3 by 3 Example

$$\mathbf{X}' = A\mathbf{X} \text{ where } A = \begin{pmatrix} 4 & 4 & -11 \\ -16 & -1 & 14 \\ 9 & -6 & -6 \end{pmatrix}$$

Characteristic Polynomial is $\det(A - \lambda I)$

$$= \lambda^3 + 3\lambda^2 + 225\lambda + 675$$
$$= \lambda^3 + 225\lambda + 3\lambda^2 + 675$$
$$= \lambda(\lambda^2 + 225) + 3(\lambda^2 + 225)$$
$$= (\lambda + 3)(\lambda^2 + 225)$$

So eigenvalues are $\lambda = -3, \lambda = \pm 15i$

EIGENVECTORS: $(A-\lambda I)\mathbf{v} = \mathbf{0} = (A-\lambda I)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{0}$

$$A = \begin{pmatrix} 4 & 4 & -11 \\ -16 & -1 & 14 \\ 9 & -6 & -6 \end{pmatrix}$$

For $\lambda = -3, A - \lambda I = A + 3I = \begin{pmatrix} 7 & 4 & -11 \\ -16 & 2 & 14 \\ 9 & -6 & -3 \end{pmatrix}$

which row reduces to

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -0 & 0 \end{pmatrix} \text{ so } \begin{matrix} v_1 = v_3 \\ v_2 = v_3 \\ v_3 \text{ any value} \end{matrix} \quad \text{Take } \mathbf{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence one solution to our system of differential equations is

$$e^{-3t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

EIGENVECTOR FOR $\lambda = 15i$

Here $$A - \lambda I = A - 15i = \begin{pmatrix} 4-15i & 4 & -11-15i \\ -16 & -1 & 14 \\ 9 & -6- & -615i \end{pmatrix}$$

which row reduces to

$$\begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & 1+i \\ 0 & 0 & 0 \end{pmatrix} \text{ so } \begin{matrix} w_1 = iw_3 \\ w_2 = -(1+i)w_3 \\ w_3 \text{ any value} \end{matrix} \text{ Take } \mathbf{w} = \begin{pmatrix} i \\ -1-i \\ 1 \end{pmatrix}$$

We can write $\mathbf{w}$ as $$\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \mathbf{p} + i\mathbf{r}$$

Solutions Associated with $\lambda = 15i$

Eigenvector: $\mathbf{w} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \mathbf{p} + i\mathbf{r}$

Solution: $e^{15it}(\mathbf{p} + i\mathbf{r}) = (\cos 15t + i \sin 15t)(\mathbf{p} + i\mathbf{r})$

$= (\cos 15t)\mathbf{p} + i(\cos 15t)i\mathbf{r} + i(\sin 15t)\mathbf{p} + i^2(\sin 15t)\mathbf{r}$

$=[(\cos 15t)\mathbf{p} - (\sin 15t)\mathbf{r}] + i\,[\cos 15t)\mathbf{r} + (\sin 15t)\mathbf{p}]$

Each term in square brackets is itself as solution.

The first is $(\cos 15t) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - (\sin 15t) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

which equals $\begin{pmatrix} -\sin 15t \\ \sin 15t - \cos 15t \\ \cos 15t \end{pmatrix}$

Similarly, the second is $\begin{pmatrix} \cos 15t \\ -\sin 15t - \cos 15t \\ \sin 15t \end{pmatrix}$

The General Solution to $\mathbf{X}' = A\mathbf{X}$ where $A = \begin{pmatrix} 4 & 4 & -11 \\ -16 & -1 & 14 \\ 9 & -6 & -6 \end{pmatrix}$:

$$C_1 e^{-3t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} -\sin 15t \\ \sin 15t - \cos 15t \\ \cos 15t \end{pmatrix} + C_3 \begin{pmatrix} \cos 15t \\ -\sin 15t - \cos 15t \\ \sin 15t \end{pmatrix}$$