MATH 226: Differential Equations

Class 21: Friday, April 4 2025

Handouts

Notes on Assignment 13

Assignment 14

Announcements

Project 2 Due Monday
Exam 2 Wednesday, April16

Theorem: Let $\lambda_1, \lambda_2, ..., \lambda_m$ be m **distinct** eigenvalues of a square matrix A with corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}$, respectively; that is, $A\mathbf{v_i} = \lambda_i \mathbf{v_i}$ for $i = 1, 2, 3, ..., m$. Then the set $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_m}\}$ is linearly independent.

Consequently, the functions $e^{\lambda_1 t}\mathbf{v_1}, e^{\lambda_2 t}\mathbf{v_2}, ... e^{\lambda_m t}\mathbf{v_m}$ form a linearly independent set of solutions to the system $\mathbf{x}' = A\mathbf{x}$.

Proof of the Theorem via *Mathematical Induction*

. We have proved the cases $m = 1$ and $m = 2$

. Suppose the Theorem is true for some positive integer $m = k$.

. Now consider $\lambda_1, \lambda_2, ..., \lambda_k, \lambda_{k+1}$ be $k + 1$ **distinct** eigenvalues of a square matrix A with corresponding eigenvectors $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}, \mathbf{v_{k+1}}$, respectively.

Consider a linear combination of these $k + 1$ vectors equal to $\mathbf{0}$:

. (*) $C_1\mathbf{v_1} + C_2\mathbf{v_2} + ... + C_k\mathbf{v_k} + C_{k+1}\mathbf{v_{k+1}} = \mathbf{0}$

Multiply (*) by A and then by λ_{k+1} to obtain

. (**) $C_1\lambda_1\mathbf{v_1} + C_2\lambda_2\mathbf{v_2} + ... + C_k\lambda_k\mathbf{v_k} + C_{k+1}\lambda_{k+1}\mathbf{v_{k+1}} = \mathbf{0}$

. (***)

$$C_1\lambda_{k+1}\mathbf{v_1} + C_2\lambda_{k+1}\mathbf{v_2} + ... + C_k\lambda_{k+1}\mathbf{v_k} + C_{k+1}\lambda_{k+1}\mathbf{v_{k+1}} = \mathbf{0}$$

. Subtract (***) from (**)

. $C_1(\lambda_1 - \lambda_{k+1})\mathbf{v_1} + C_2(\lambda_2 - \lambda_{k+1})\mathbf{v_2} + ... + C_k(\lambda_k - \lambda_{k+1})\mathbf{v_k} = \mathbf{0}$

. But $\{\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_k}\}$ is a linearly independent set.

. Hence each $C_i(\lambda_i - \lambda_{k+1})$ is 0. These means each C_i is 0 for $i = 1, 2, ..., k$.

. Substitute back into (*) to obtain $C_{k+1}\mathbf{v_{k+1}} = \mathbf{0}$, implying C_{k+1} also $= 0$.

Example 1

$$A = \begin{bmatrix} -40 & -6 & 19 & -28 & 50 \\ -69 & -13 & 35 & -44 & 88 \\ 114 & 26 & -53 & 56 & -138 \\ 1 & 2 & -1 & -1 & -2 \\ -87 & -16 & 41 & -52 & 107 \end{bmatrix}$$

Characteristic Polynomial:

$$\lambda^5 - 20\lambda^3 + 30\lambda^2 + 19\lambda - 30$$

$$= (\lambda + 5)(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda + 1)$$

Eigenvalue	Eigenvector (written horizontally)
3	$\mathbf{v}_1 =$ (2,4,2,1,2)
1	$\mathbf{v}_2 =$ (1,2,-1,1,2)
-1	$\mathbf{v}_3 =$ (2,1,-2,1,3)
2	$\mathbf{v}_4 =$ (1,3,2,1,1)
-5	$\mathbf{v}_5 =$ 1,-1,3,1,0)

$$C_1 e^{3t}\mathbf{v_1} + C_2 e^{t}\mathbf{v_2} + C_3 e^{-t}\mathbf{v_3} + C_4 e^{2t}\mathbf{v_4} + C_5 e^{-5t}\mathbf{v_5}$$

The Matrix A Is Nondefective With Real Eigenvalues

Let $(\lambda_1, \mathbf{v}_1), \ldots, (\lambda_n, \mathbf{v}_n)$ be eigenpairs for the real, $n \times n$ constant matrix $\mathbf{A}$. Assume that the eigenvalues $\lambda_1, \ldots, \lambda_n$ are real and that the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Then

$$\left\{e^{\lambda_1 t}\mathbf{v}_1, \ldots, e^{\lambda_n t}\mathbf{v}_n\right\}$$

is a fundamental set of solutions to $\mathbf{x}' = \mathbf{A}\mathbf{x}$ on the interval $(-\infty, \infty)$. The general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is therefore given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{v}_n,$$

where $c_1, \ldots, c_n$ are arbitrary constants.

Example 2

$$A = \begin{bmatrix} 8 & 1 & -3 & 4 & -7 \\ -21 & -2 & 9 & -10 & 25 \\ -6 & -1 & 5 & -4 & 7 \\ -5 & -1 & 2 & 0 & 6 \\ 1 & 0 & -1 & 2 & 1 \end{bmatrix}$$

Characteristic Polynomial:

$$\lambda^5 - 12\lambda^4 + 57\lambda^3 - 134\lambda^2 + 156\lambda - 72$$
$$= (\lambda - 3)^2(\lambda - 2)^3$$

$\lambda = 2$ has algebraic multiplicity 3 and geometric multiplicity 3 with a linearly independent set of 3 vectors

$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$

$=$ { (1,1,0,0,1), (-2,8,0,1,0), 1,-3,1,0,0) }

$\lambda = 3$ has algebraic multiplicity 2 and geometric multiplicity 2 with a linearly independent set of 2 vectors

$\{\mathbf{w_1}, \mathbf{w_2}\} =$ { (1,-1,-1,0,1), (-1,4,1,1,0) }

The General Solution to $\mathbf{X}' = A\mathbf{X}$ is

$$C_1 e^{2t}\mathbf{v_1} + C_2 e^{2t}\mathbf{v_2} + C_3 e^{2t}\mathbf{v_3} + C_4 e^{3t}\mathbf{w_1} + C_5 e^{3t}\mathbf{w_2}$$


I'VE COVERED EVERY CONTINGENCY. WHAT COULD POSSIBLY GO WRONG?
HEY, EVEN I DON'T SAY THAT OUT LOUD.


DEFECTIVE
MATRIX

Example 4

$$A = \begin{bmatrix} 39 & 7 & -17 & 23 & -45 \\ -26 & -4 & 13 & -15 & 34 \\ -83 & -17 & 42 & -52 & 104 \\ 11 & 2 & -5 & 9 & -13 \\ 62 & 12 & -29 & 39 & -74 \end{bmatrix}$$

Characteristic Polynomial:

$$\lambda^5 - 12\lambda^4 + 57\lambda^3 - 134\lambda^2 + 156\lambda - 72$$
$$= (\lambda - 3)^2(\lambda - 2)^3$$

$\lambda = 2$ has algebraic multiplicity 3 but geometric multiplicity 1 with only 1 linearly independent eigenvector (0,4,3,1,0)

$\lambda = 3$ has algebraic multiplicity 2 but geometric multiplicity 1 with only 1 linearly independent eigenvector (1,2,-1,1,2)

Example 5

$$A = \begin{bmatrix} 19 & 3 & -8 & 10 & -20 \\ 1 & 2 & -1 & 2 & -1 \\ -17 & -3 & 10 & -10 & 20 \\ 6 & 1 & -3 & 6 & -7 \\ 23 & 4 & -11 & 14 & -25 \end{bmatrix}$$

Characteristic Polynomial:

$$\lambda^5 - 12\lambda^4 + 57\lambda^3 - 134\lambda^2 + 156\lambda - 72$$
$$= (\lambda - 3)^2(\lambda - 2)^3$$

$\lambda = 2$ has algebraic multiplicity 3 **and** geometric multiplicity 3 with a linearly independent set of 3 vectors

$$\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$$

= { (1,1,0,0,1), (-2,8,0,1,0), 1,-3,1,0,0) }

$\lambda = 3$ has algebraic multiplicity 2 **but** geometric multiplicity 1 with only 1 linearly independent eigenvector (1,2,-1,1,2)

A is 5×5 matrix so solving $\mathbf{X}' = A\mathbf{X}$ involves finding 5 linearly independent solutions. We have 4.

How do we find a 5th?

$\lambda = 3$ has algebraic multiplicity 2 **but** geometric multiplicity 1 with only 1 linearly independent eigenvector $\mathbf{v} = (1, 2, -1, 1, 2)$.

Recall what we did in 2×2 case

We formed a new solution of the form $te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$ where $\mathbf{w}$ was chosen so that $(A - \lambda I)\mathbf{w} = \mathbf{v}$ so that $A\mathbf{w} - \lambda\mathbf{w} = \mathbf{v}$ or

$$A\mathbf{w} = \lambda\mathbf{w} + \mathbf{v}$$

We can do the same thing here:

$$(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w})' = t\lambda e^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}\mathbf{w}$$

$$A(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}) = te^{\lambda t}A\mathbf{v} + e^{\lambda t}A\mathbf{w}$$

$$= te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}\lambda\mathbf{w} + e^{\lambda t}\mathbf{v}$$

Some Conditions To Check:

The vectors $\mathbf{v}$ and $\mathbf{w}$ form a Linearly Independent Set

The Five Solutions Form a Linearly Independent Set of Functions

Return to **Example 4**

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Characteristic Polynomial:

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$$= (\lambda - 3)^2(\lambda - 2)^3$$

$\lambda = 2$ **has algebraic multiplicity 3 but geometric multiplicity 1 with only 1 linearly independent eigenvector (0,4,3,1,0)**

$\lambda = 3$ has algebraic multiplicity 2 but geometric multiplicity 1 with only 1 linearly independent eigenvector (1,2,-1,1,2)

$\lambda = 2$ **has algebraic multiplicity 3 but geometric multiplicity 1 with only 1 linearly independent eigenvector (0,4,3,1,0)**

We are short 2 solutions

Suppose $\mathbf{v}$ is an eigenvalue associated with λ

and $\mathbf{w}$ satisfies $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Then $e^{\lambda t}\mathbf{v}$ and $te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$ are solutions.

To find a third:

Pick vector $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Then $\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$ is also a solution.

Suppose λ **has algebraic multiplicity 4 but geometric multiplicity 1 with only 1 linearly independent eigenvector v**

Pick vectors $\mathbf{w}, \mathbf{u}, \mathbf{s}$ so that

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$
$$(A - \lambda I)\mathbf{u} = \mathbf{w}$$
$$(A - \lambda I)\mathbf{s} = \mathbf{u}$$

Then 4 solutions are

$$e^{\lambda t}\mathbf{v}$$
$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$
$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$
$$\frac{t^3}{3!}e^{\lambda t}\mathbf{v} + \frac{t^2}{2!}e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{s}$$

What's Next?

$x' = ax$ has solution $x = Ce^{at}$

Could $\mathbf{X}' = AX$ have solution $\mathbf{X} = Ce^{At}$?