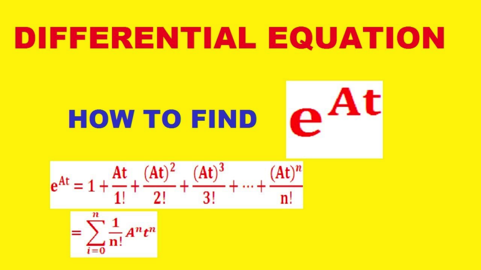
MATH 226: Differential Equations



Class 22: April 7, 2025

Notes on Assignment 13
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Notes on Sample Exam 2
Matrix Exponential Power Series
Computing Matrix Exponential From a Fundamental Matriix

Announcements

Exam 2: Wednesday, November 16

Defective Matrices

Suppose λ is an eigenvalue of an $n \times n$ Matrix A with algebraic multiplicity 3 and geometric multiplicity 1.

Let **v** be an eigenvector associated with λ and **w** and **u** are vectors such that

$$(A - \lambda I)\mathbf{w} = \mathbf{v}$$

$$(A - \lambda I)\mathbf{u} = \mathbf{w}$$

. Then

- (1) $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ is a linearly independent set of vectors
- (2) Each of $e^{\lambda t}\mathbf{v}$, $te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$, and $\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$, is a solution of $\mathbf{X'} = A\mathbf{X}$.
- (3) $\{e^{\lambda t}\mathbf{v}, te^{\lambda t}v + e^{\lambda t}\mathbf{w}, \frac{t^2 2}{e}^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}\}$ is a linearly independent set of solutions.

$\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ is a linearly independent set of vectors.

Proof:

Use $(A - \lambda I)\mathbf{v} = 0, (A - \lambda I)\mathbf{w} = \mathbf{v}, (A - \lambda I)\mathbf{u} = \mathbf{w}$

Suppose (*) $C_1\mathbf{v} + C_2\mathbf{w}, +C_3\mathbf{u} = \mathbf{0}$ for some constants C_1, C_2, C_3

Multiply (*) by $(A - \lambda I)$:

$C_1(A - \lambda I)\mathbf{v} + C_2(A - \lambda I)\mathbf{w}, +C_3(A - \lambda I)\mathbf{u} = (A - \lambda I)\mathbf{0}$

(**): $0 + C_2\mathbf{v}, +C_3\mathbf{w}, = \mathbf{0}$

Multiply (**) by $(A - \lambda I)$: $C_3\mathbf{v} = \mathbf{0}$ Thus $C_3 = 0$

(**) becomes $C_2\mathbf{v} = 0$

Thus $C_2 = 0$

Then (*) becomes $C_1\mathbf{v} = 0$ and hence $C_1 = 0$.

Each of $e^{\lambda t}\mathbf{v}$, $te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$, and $\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$, is a solution of

Proof:

From $(A - \lambda I)\mathbf{v} = 0, (A - \lambda I)\mathbf{w} = \mathbf{v}, (A - \lambda I)\mathbf{u} = \mathbf{w}$

We have $A\mathbf{v} = \lambda\mathbf{v}$

X' $= A\mathbf{X}$. $A\mathbf{w} = \mathbf{v} + \lambda\mathbf{w}$

$A\mathbf{u} = \mathbf{w} + \lambda\mathbf{u}$

We'll show $\mathbf{X} = te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$ is a solution of **X'** $= A\mathbf{X}$.

X' $= e^{\lambda t}\mathbf{v} + t\lambda e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}\mathbf{w}$

while $A\mathbf{X} = te^{\lambda t}A\mathbf{v} + e^{\lambda t}A\mathbf{w} = te^{\lambda t}\lambda\mathbf{v} + e^{\lambda t}\mathbf{v} + \lambda e^{\lambda t}\mathbf{w}$

$\{e^{\lambda t}\mathbf{v}, te^{\lambda t}v + e^{\lambda t}\mathbf{w}, \frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}\}$ is a linearly independent set of solutions.

Proof: Supppose

$$C_1 e^{\lambda t}\mathbf{v} + C_2(te^{\lambda t}v + e^{\lambda t}\mathbf{w}) + C_3(\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}) = 0$$

Evaluate at $t = 0$

$$C_1 1\mathbf{v} + C_2(0v + 1\mathbf{w}) + C_3(0\mathbf{v} + 0\mathbf{w} + 1\mathbf{u}) = 0$$

$$C_1\mathbf{v} + C_2\mathbf{w} + C_3\mathbf{u} = 0$$

Major Goal:

Study Systems of First Order Linear Differential Equations

$$\mathbf{x}' = \mathbf{P}(t)\,\mathbf{x} + \mathbf{g}(t)$$

where $\mathbf{P}(t)$ is a square matrix of continuous functions and $\mathbf{g}(t)$ is a vector of continuous functions.

Special Case: $\mathbf{x}' = \mathbf{A}\,\mathbf{x}$

where $\mathbf{A}$ is a square matrix of constants.

For each eigenvector $\mathbf{v}$ associated with an eigenvalue λ of the matrix $\mathbf{A}$:

$$e^{\lambda t}\mathbf{v}$$

is a solution.

Matrix Exponential Function

Our very first example in the course

$x' = ax$ where a is a constant

has a solution of the form $x = e^{at}$

By analogy, $\mathbf{X'} = A\,\mathbf{X}$

"ought" to have a solution of the form

$\mathbf{X} = e^{At}$

But What is the Exponential of a Matrix?

Applying Exponential Function to a Matrix

Recall that the square of a matrix is not the matrix of squares.

So we don't expect to get the matrix of exponentials.

Begin with the expression for e^{at} as a power series:

$$e^{at} = 1 + at + \frac{a^2 t^2}{2!} + \frac{a^3 t^3}{3!} + \frac{a^4 t^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{a^k t^k}{k!}$$

This series converges absolutely for all a and t.
Let's define e^{At} as

$$e^{At} = I + At + \frac{A^2 t^2}{2!} + \frac{A^3 t^3}{3!} + \frac{A^4 t^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

We can compute each term in this series; it will be an $n \times n$ matrix if A is an $n \times n$ matrix.

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Some Properties of e^{At}:

1. $e^{A0} = e^0 = I$
2. $(e^{At})' = 0 + A + \frac{A^2}{2!}2t + \frac{A^3}{3!}3t^2 + \frac{A^4}{4!}4t^3 \ldots +$
   $= A + A^2t + \frac{A^3}{2!}t^2 + \frac{A^4}{3!}t^3 + \ldots$
   $= A(I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \ldots)$
   $= Ae^{At}$
   so e^{At} is a solution of $\mathbf{x}' = A\mathbf{x}$.
3. Each column is a solution
4. Columns are linearly independent
5. $e^{-At} = (e^{At})^{-1}$ (Matrix Inverse)
6. $e^{A(s+t)} = e^{As}e^{At}$
7. $e^{(A+B)t} = e^{At}e^{Bt}$ if $AB = BA$
8. $Ae^{At} = e^{At}A$

Computing e^{At} via Power Series

Need $A, A^2, A^3, A^4, \ldots$

Note: $A^2 = AA, A^3 = AA^2, A^4 = AA^3, etc$

Example $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}$$

$$A^6 = \begin{pmatrix} 5743 & 8370 \\ 12555 & 18298 \end{pmatrix}$$

Not so easy to see what e^{At} actually looks like!

Try a different example: $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

[See *Maple* "Matrix Exponential Power Series"]

$$\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ -(a+2c) & -(b+2d) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}, A^3 = \begin{pmatrix} 2 & 3 \\ -3 & -4 \end{pmatrix}, A^4 = \begin{pmatrix} -3 & -4 \\ 4 & 5 \end{pmatrix}$$

A^k

k even	k odd
$\begin{pmatrix} -(k-1) & -k \\ k & k+1 \end{pmatrix}$	$\begin{pmatrix} k-1 & k \\ -k & -(k+1) \end{pmatrix}$

General Formula: $A^k = \begin{pmatrix} (-1)^{k+1}(k-1) & (-1)^{k+1}k \\ (-1)^k k & (-1)^k(k+1) \end{pmatrix}$

Our Example: $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

Found: $A^k = \begin{pmatrix} (-1)^{k+1}(k-1) & (-1)^{k+1}k \\ (-1)^k k & (-1)^k(k+1) \end{pmatrix}$

Upper Left: 0,-1, 2,-3,4,-5, . . .

Upper Right: 1, -2, 3, -4, 5,. . .

Lower Left: -1, 2,-3,4,-5, . . .

Lower Right: -2, 3, -4, 5, . . .

Power Series:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \frac{A^4t^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

Examine Upper Right Entries:

$$0 + 1t - 2\frac{t^2}{2!} + 3\frac{t^3}{3!} - 4\frac{t^4}{4!} + 5\frac{t^5}{5!} - 6\frac{t^6}{6!} + \ldots$$

$$= t\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right] = te^{-t}$$

Work out other three entries:

$$e^{At} = \begin{pmatrix} e^{-t}(t+1) & te^{-t} \\ -te^{-t} & (1-t)e^{-t} \end{pmatrix}$$

THERE'S GOT TO BE A BETTER WAY!

An Alternative Way To Compute e^{at}

Idea: Let Φ be any fundamental solution matrix for $\mathbf{x}' = A\mathbf{x}$ with $\Phi(0) = I$.

Then $\Phi(t)$ and e^{At} are both solutions of $\mathbf{x}' = A\mathbf{x}$ which satisfy the same initial condition.

The Uniqueness of Solutions Theorem implies $\Phi(t) \equiv e^{At}$

How to find Φ.

1. Use Eigenvalue/ Eigenvector approach to find a full linearly independent set of solutions to X' = Ax.
2. Enter them as columns in a matrix in a matrix X(t) (This is a fundamental matrix)
3. $X(t)$ is invertible for all t. Thus X(0) is an invertible matrix of constants
4. Define $\Phi(t) = X(t)\,[X(0)]^{-1}$

$$\text{Then } \Phi(0) = X(0)\,[X(0)]^{-1} = I$$

$$\text{and } \Phi'(t) = X'(t)\,[X(0)]^{-1} = AX(t)\,[X(0)]^{-1} = A\Phi(t)$$

so Φ is also a solution.

Using this approach for $\mathbf{x}' = A\mathbf{x}$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{pmatrix} = 2\lambda + \lambda^2 + 1 = (\lambda+1)^2$$

$\lambda = -1$ is double root; algebraic multiplicity $= 2$

To find eigenvectors: $(A - \lambda I) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ which reduces to

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \text{ so } \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Geometric Multiplicity $= 1$.

One solution is $e^{-1t}\mathbf{v} = e^{-}t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

To find another, solve $(A - \lambda I)\mathbf{w} = \mathbf{v}$ for $\mathbf{w}$

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ implies } \begin{array}{r} w_1 + w_2 = 1 \\ -w_1 - w_2 = -1 \end{array}$$

Choose $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Solutions are $e^{-t}\mathbf{v}$, $te^{-t}\mathbf{v} + e^{-t}\mathbf{w}$

$$\mathbf{x_1} = e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$\mathbf{x_2} = te^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} te^{-t} + e^{-t} \\ -te^{-t} \end{pmatrix} = \begin{pmatrix} e^{-t}(t+1) \\ e^{-t}(-t) \end{pmatrix}$$

$$\mathbf{X}(t) = \begin{pmatrix} e^{-t} & e^{-t}(t+1) \\ -e^{-t} & e^{-t}(-t) \end{pmatrix} \text{ so } \mathbf{X}(\mathbf{0}) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\text{Then } [\mathbf{X}(\mathbf{0})]^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Thus

$$e^{At} = \Phi(t) = \mathbf{X}(t)[\mathbf{X}(\mathbf{0})]^{-1} = \begin{pmatrix} e^{-t} & e^{-t}(t+1) \\ -e^{-t} & e^{-t}(-t) \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t}(t+1) & te^{-t} \\ -e^{-t}t & e^{-t}(1-t) \end{pmatrix}$$

Review

The Matrix Exponential

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \ldots + A^k\frac{t^k}{k!} + \ldots$$

e^{At} is an $n \times n$ matrix

Each column of e^{At} is a solution of **x'** $= A\mathbf{x}$

The columns form a linearly independent set

Some Other Nice Properties:

$$e^{A\times 0} = I$$

$$(e^{At})' = Ae^{At}$$

$$e^{-At} = (e^{At})^{-1}$$

$$e^{A(r+s)} = e^{Ar}e^{As}$$

Review

e^{At} **has wonderful properties but it is hard to compute via the power series definition.**

Alternate Way To Compute Matrix Exponential e^{At}

$$e^{At} = X(t)(X(0))^{-1}$$

where $X(t)$ **is any Fundamental Matrix for x' $= A$x.**

How To Find X?
Use eigenvalue/eigenvector approach.

Nonhomogeneous Systems

Recall Solution of $x' = ax + g(t)$

$$x' - ax = g(t)$$

Multiply by integrating factor e^{-at}

$$(xe^{-at})' = e^{-at}g(t)$$

$$xe^{-at} = \int e^{-at}g(t)\,dt + C$$

$$x = e^{at}\int e^{-at}g(t)\,dt + Ce^{at}$$

$$x = e^{at}\int_0^t e^{-as}g(s)\,ds + Ce^{at}$$

Evaluate at $t = 0$:

$$x = e^{at}\int_0^t e^{-as}g(s)\,ds + x(0)e^{at}$$

Nonhomogeneous Systems

$x' = ax + g(t)$ has solution $x = e^{at} \int_0^t e^{-as} g(s)\, ds + e^{at} x(0)$

X' $= A\mathbf{X} + \mathbf{g}(t)$ has solution

$$\mathbf{X} = e^{At} \int_0^t e^{-As} \mathbf{g}(s) + e^{At}\mathbf{X}(0)$$

$$\mathbf{X} = \Phi(t) \int_0^t \Phi^{-1}(s)\mathbf{g}(s) + \Phi(t)\mathbf{X}(0)$$

We can also write the solution as

$$\mathbf{X}(t) \int_{t_0}^t \mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds + \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$$

where **X** is any fundamental solution of **X'** $= A\mathbf{X}$