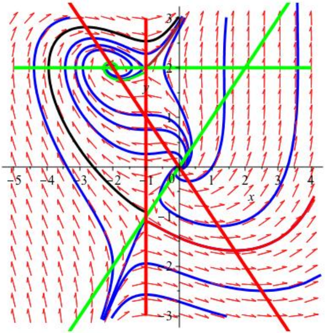
MATH 226: Differential Equations



Class 23: Wednesday, April 9, 2025

Handouts

Variation of Parameters 1

Variation of Parameters 2

Announcements

Student Symposium: Friday, April 11
Assignment 15: Due Monday, April 14

Exam 2: Wednesday Evening, April 16

Department of Mathematics and Statistics

Pre-registration Dessert Social

Monday, 4/14 | 3:00-4:30pm | Warner 105

Interested in taking some Math or Stat courses in **Fall 2025**? Currently taking a Math or Stats class? Need a study break?

Join the Math & Stats faculty over dessert to:

- Learn about Fall 2025 course offerings
- Get information about:
  - Major in Mathematics and/or the Applied Math Track
  - Major in Statistics
  - Minor in Mathematics
- Ask questions and receive advice about how Math and Stats fits into your Middlebury experience
- Be in community and hear from other students about Math and Stat courses

Anyone who is currently taking or wants to take a Math or Stats course is welcome! Even if you're graduating in May, we hope to see you at the dessert social!

A special note to our Math & Stat Majors: we will take class year

Today's Agenda

- Nonhomogenous Systems
- More on Defective Matrices
- Quick Peek at Nonlinear Systems

Nonhomogeneous Systems

Recall Solution of $x' = ax + g(t)$

$$x' - ax = g(t)$$

Multiply by integrating factor e^{-at}

$$(xe^{-at})' = e^{-at} g(t)$$

$$xe^{-at} = \int e^{-at} g(t)\, dt + C$$

$$x = e^{at} \int e^{-at} g(t)\, dt + Ce^{at}$$

$$x = e^{at} \int_0^t e^{-as} g(s)\, ds + Ce^{at}$$

Evaluate at $t = 0$:

$$x = e^{at} \int_0^t e^{-as} g(s)\, ds + x(0)e^{at}$$

Nonhomogeneous Systems

$x' = ax + g(t)$ has solution $x = e^{at}\int_0^t e^{-as} g(s)\, ds + e^{at} x(0)$

$\mathbf{X'} = A\mathbf{X} + \mathbf{g}(t)$ has solution

$$\mathbf{X} = e^{At}\int_0^t e^{-As}\mathbf{g}(s) + e^{At}\mathbf{X}(0)$$

$$\mathbf{X} = \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{g}(s) + \Phi(t)\mathbf{X}(0)$$

We can also write the solution as

$$\mathbf{X}(t)\int_{t_0}^t \mathbf{X}^{-1}(s)\mathbf{g}(s)\, ds + \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$$

where $\mathbf{X}$ is any fundamental solution of $\mathbf{X'} = A\mathbf{X}$

More on Defective Matrices

Solving **x'** = A**x** when A is "defective"

Suppose λ is an eigenvalue of A with algebraic multiplicity 3 but geometric multiplicity 1.

Find **v** such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$

Find **w** such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$

Find **u** such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Then 3 linearly independent solutions of **x'** = A**x** are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

What to do if λ is an eigenvalue of A with algebraic multiplicity **bigger than 3** but geometric multiplicity 1?

WHY DOES THIS WORK?

KEY STEP:

$$(A - \lambda I)\mathbf{v} = \mathbf{0} \text{ implies } A\mathbf{v} = \lambda \mathbf{v}$$
$$(A - \lambda I)\mathbf{w} = \mathbf{v} \text{ implies } A\mathbf{w} = \mathbf{v} + \lambda \mathbf{w}$$
$$(A - \lambda I)\mathbf{u} = \mathbf{w} \text{ implies } A\mathbf{u} = \mathbf{w} + \lambda \mathbf{u}$$

Find $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$
Find $\mathbf{w}$ such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$
Find $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$

Note: 3 systems of linear algebraic equations with the **same coefficient matrix**

<u>Example</u> $\begin{pmatrix} 4 & 8 & 12 \\ -82 & 204 & 295 \\ 64 & -128 & -184 \end{pmatrix}$

Characteristic Polynomial : $\lambda^3 - 24\lambda^2 + 192\lambda - 512 = (\lambda - 8)^3$

Eigenvalue $\lambda = 8$ has algebraic multiplicity 3, but geometric multiplicity only 1.

Here $(A - \lambda I) = \begin{pmatrix} -4 & 8 & 12 \\ -82 & 196 & 295 \\ 64 & -128 & -192 \end{pmatrix}$

To solve $(A - \lambda I)\mathbf{x} = \mathbf{b}$ construct augmented matrix

$$\left(\begin{array}{ccc|c} -4 & 8 & 12 & a \\ -82 & 196 & 295 & b \\ 64 & -128 & -192 & c \end{array}\right)$$

and reduce to row echelon form

To solve $(A-\lambda I)\mathbf{x}=\mathbf{b}$:

$$\text{Augmented matrix is } \begin{pmatrix} -4 & 8 & 12 & | \; a \\ -82 & 196 & 295 & | \; b \\ 64 & -128 & -192 & | \; c \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{16} & | \; \frac{49}{32}a+\frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & | -\frac{41}{64}a+\frac{1}{32}b \\ 0 & 0 & 0 & | \; 16a+c \end{pmatrix} \text{ so } \mathbf{x}=\begin{pmatrix} \frac{49}{32}a+\frac{1}{16}b-\frac{1}{16}x_3 \\ -\frac{41}{64}a+\frac{1}{32}b-\frac{49}{32}x_3 \\ 16a+c \end{pmatrix}$$

For $(A-\lambda I)\mathbf{v}=\mathbf{0}$, set $a=0, b=0, c=0$:

$$\mathbf{v}=\begin{pmatrix} -(1/16)x_3 \\ -(49/32)x_3 \\ x_3 \end{pmatrix}=x_3\begin{pmatrix} -1/16 \\ -49/32 \\ 1 \end{pmatrix}=\begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix} \text{ if } x_3=32$$

$$\begin{pmatrix} 1 & 0 & \frac{1}{16} & \mid \frac{49}{32}a + \frac{1}{16}b \\ 0 & 1 & \frac{49}{32} & \mid -\frac{41}{64}a + \frac{1}{32}b \\ 0 & 0 & 0 & \mid 16a + c \end{pmatrix} \text{ so } \mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ 16a + c \end{pmatrix}$$

For $(A - \lambda I)\mathbf{w} = \mathbf{v}$, set $a = -2, b = -49, c = 32$:

$$\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} (-1/16)w_3 \\ 1/4 - (49/32)w_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix} \text{ if } w_3 = 0.$$

$$\mathbf{x} = \begin{pmatrix} \frac{49}{32}a + \frac{1}{16}b - \frac{1}{16}x_3 \\ \\ -\frac{41}{64}a + \frac{1}{32}b - \frac{49}{32}x_3 \\ \\ 16a + c \end{pmatrix}$$

For $(A - \lambda I)\mathbf{u} = \mathbf{w}$, set $a = 0, b = -1/4, c = 0$:

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} (-1/64) - (1/16)u_3 \\ (-1/128) - (49/32)u_3 \\ u_3 \end{pmatrix} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix} \text{ if } u_3 = 0.$$

Thus $\mathbf{v} = \begin{pmatrix} -2 \\ -49 \\ 32 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 1/4 \\ 0 \end{pmatrix}$, $\mathbf{u} = \begin{pmatrix} -1/64 \\ -1/128 \\ 0 \end{pmatrix}$

Solving **x'** $=$ A**x** when A is "defective"

Suppose λ is an eigenvalue of A with algebraic multiplicity 4 but geometric multiplicity 1.

Find $\mathbf{v}$ such that $(A - \lambda I)\mathbf{v} = \mathbf{0}$
Find $\mathbf{w}$ such that $(A - \lambda I)\mathbf{w} = \mathbf{v}$
Find $\mathbf{u}$ such that $(A - \lambda I)\mathbf{u} = \mathbf{w}$
Find $\mathbf{z}$ such that $(A - \lambda I)\mathbf{z} = \mathbf{u}$

Then 4 linearly independent solutions of **x'** $=$ A**x** are

$$e^{\lambda t}\mathbf{v}$$

$$te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{w}$$

$$\frac{t^2}{2}e^{\lambda t}\mathbf{v} + te^{\lambda t}\mathbf{w} + e^{\lambda t}\mathbf{u}$$

$$\frac{t^3}{3!}e^{\lambda t}\mathbf{v} + \frac{t^2}{2}e^{\lambda t}\mathbf{w} + te^{\lambda t}\mathbf{u} + e^{\lambda t}\mathbf{z}$$

Next Major Goal:

Study Nonlinear Systems
of General First Order Differential Equations

$$\begin{aligned} x' &= F(x,y,t) \\ y' &= G(x,y,t) \end{aligned} \qquad \begin{aligned} x' &= F(x,y,z,t) \\ y' &= G(x,y,z,t) \\ z' &= H(x,y,z,t) \end{aligned}$$

General Case

$$\begin{aligned} x_1' &= f_1(x_1, x_2, ..., x_n, t) \\ x_2' &= f_2(x_1, x_2, ..., x_n, t) \\ &\;\vdots \\ x_n' &= f_n(x_1, x_2, ..., x_n, t) \end{aligned}$$

$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

No Explicit t on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Three Approaches:

Analytic: Rarely Possible To Find Closed Form Solution

Numeric: Detailed Information About a Single Solution

Geometric: Qualitative Information About All Solutions

Example

$$x' = (2 - y)(x - y)$$
$$y' = (1 + x)(x + y)$$

STEP ONE: Identify All Equilibrium Points

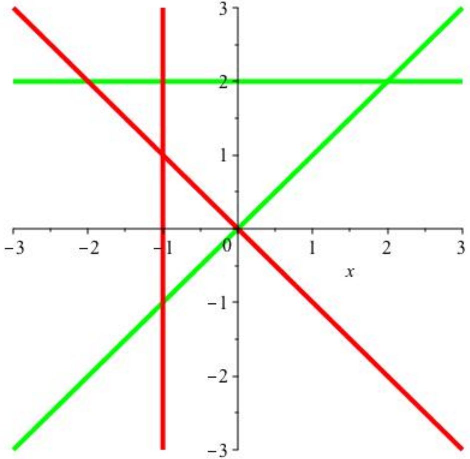
x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x

$$x' = (2-y)(x-y), y' = (1+x)(x+y)$$

x' =0 along lines y =2 and y = x

y' =0 along lines x = -1 and y = -x



$n = 2$	$n = 3$
$x' = F(x, y, t)$	$x' = F(x, y, z, t)$
$y' = G(x, y, t)$	$y' = G(x, y, z, t)$
	$z' = H(x, y, z, t)$

Autonomous Systems

No Explicit t on Right Hand Side

$n = 2$	$n = 3$
$x' = F(x, y)$	$x' = F(x, y, z)$
$y' = G(x, y)$	$y' = G(x, y, z)$
	$z' = H(x, y, z)$

Special Properties of Autonomous Systems

1. Direction Field is Independent of Time
2. Only One Trajectory Passing Through Each Point (x_0, y_0)
3. A Trajectory Can Not Cross Itself
4. A Single Well-Chosen Phase Portrait Simultaneously Displays Important Information About All Solutions