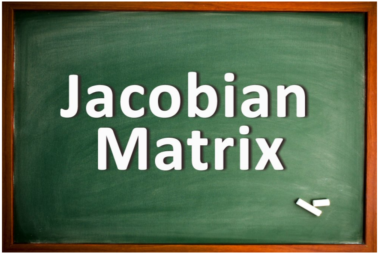
MATH 226: Differential Equations



Class 25: April 16, 2025

Maple Examples in Handouts Folder:
Simple Competition Model
Competition Model

Exam 2

Tonight

7 PM – ?

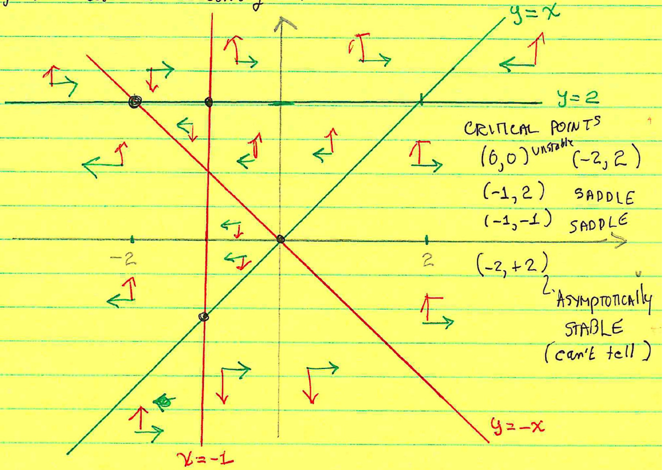
Warner ??? Office Hours Today: 12:30 – 1:30

EXAMPLE $x' = (2-y)(x-y) = 2x - 2y - xy + y^2$

$y' = (1+x)(x+y) = x + y + x^2 + xy$

$x' = 0$ at $y = 2$ and $y = x$ GREEN

$y' = 0$ at $x = -1$ and $y = -x$



$$\begin{pmatrix} x - x^* \\ y - y^* \end{pmatrix}' = \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{bmatrix} F(x,y) \\ G(x,y) \end{bmatrix} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} \begin{bmatrix} x - x^* \\ y - y^* \end{bmatrix}$$

Near Critical Point, System Behaves Like **X'** $= A$ **X** where

$$A = \begin{bmatrix} F_x(x^*, y^*) & F_y(x^*, y^*) \\ G_x(x^*, y^*) & G_y(x^*, y^*) \end{bmatrix} = J(x^*, y^*)$$

Current Goal:
Approximating Nonlinear Autonomous System with Linear System Near An Equilibrium Point

$$x' = F(x, y)$$
$$y' = G(x, y)$$

$$F(x, y) \approx F(x^*, y^*) + F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
$$G(x, y) \approx G(x^*, y^*) + G_x(x^*, y^*)(x - x^*) + G_y(x^*, y^*)(y - y^*)$$

But at Equilibrium Points, $F(x^*, y^*) = 0, G(x^*, y^*) = 0$ so

$$F(x, y) \approx F_x(x^*, y^*)(x - x^*) + F_y(x^*, y^*)(y - y^*)$$
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which we can write as

$$\begin{bmatrix} F_x(x^*,y^*) & F_y(x^*,y^*) \\ G_x(x^*,y^*) & G_y(x^*,y^*) \end{bmatrix} \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix}$$

or

$$J(x^*,y^*) \begin{bmatrix} x-x^* \\ y-y^* \end{bmatrix} = J(x^*,y^*) \begin{bmatrix} h \\ k \end{bmatrix}$$

J is called the **Jacobi Matrix** or **Jacobian**

Example $x' = (2-y)(x-y) = 2x - 2y - xy + y^2 = F(x,y)$

$y' = (1+x)(x+y) = x + y + x^2 + xy = G(x,y)$

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$$y' = (1+x)(x+y) = x + y + x^2 + xy = G(x,y)$$

$$F_x = 2 - y \qquad G_x = 1 + 2x + y$$

$$F_y = -2 - x + 2y \qquad G_y = 1 + x$$

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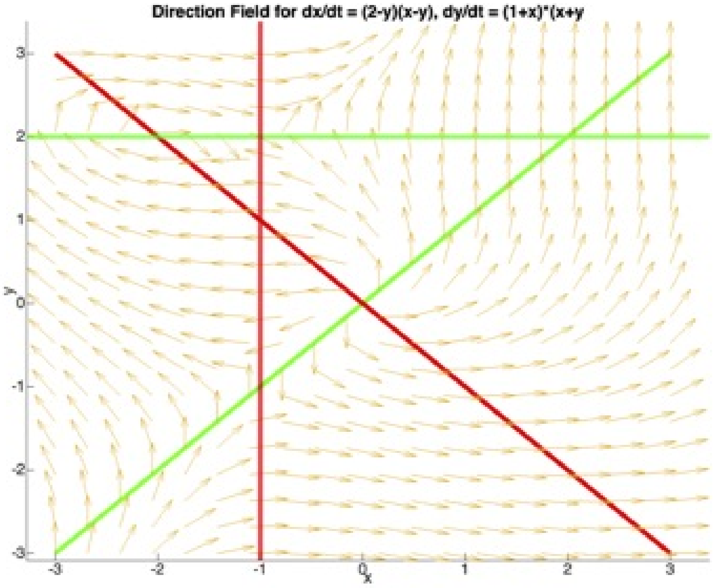
(x^*, y^*)	$J(x^*, y^*)$	Eigenvalues	Nature
(0,0)	$\begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}$	$\frac{1}{2}(3 \pm i\sqrt{7})$	Unstable Spiral
(-1,-1)	$\begin{pmatrix} 3 & -3 \\ -2 & 0 \end{pmatrix}$	$\frac{1}{2}(3 \pm \sqrt{33})$	Saddle Point
(-1,2)	$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$	$\pm\sqrt{3}$	Saddle Point
(-2,2)	$\begin{pmatrix} 0 & 4 \\ -1 & -1 \end{pmatrix}$	$\frac{1}{2}(-1 \pm i\sqrt{15})$	Asymptotically Stable Spiral

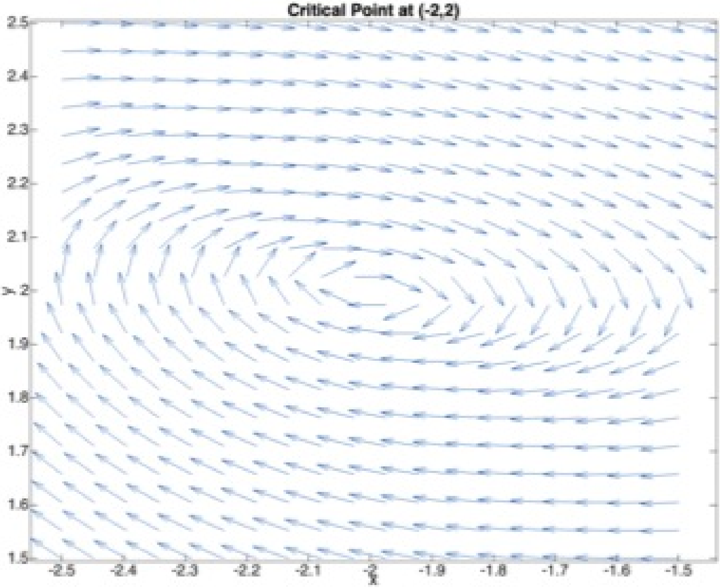
Direction Field in MATLAB

```
x = linspace(-3,3, 20);
y = linspace(-3,3,20);

hold on

plot(x,x,'g','LineWidth',3 )
yline(2,'g', 'LineWidth',3)
plot(x,-x,'r', 'LineWidth',3)
xline(-1,'r', 'LineWidth',3)

[X,Y]= meshgrid(x,y);

dXdt = (2 - Y).*(X - Y);
dYdt = (1+X).* (X + Y);

%quiver(X,Y, dXdt, dYdt, 'Linewidth', 3);
 un=dXdt./sqrt(dXdt.^2+dYdt.^2);
 wn=dYdt./sqrt(dXdt.^2+dYdt.^2);
 quiver(X,Y,un,wn)
xlabel('x')
ylabel('y')
title('Direction Field for dx/dt = (2-y)(x-y), dy/dt = (1+x)*(x+y')

axis tight;
hold off;
```


Direction Field for dx/dt = (2-y)(x-y), dy/dt = (1+x)*(x+y
3
2
1
0
-1
-2
-3
y
-3
-2
-1
0
1
2
3
x

Examine Critical Point (-2,2)

```
x = linspace(-2.5,-1.5, 20);
y = linspace(1.5,2.5,20);
[X,Y]= meshgrid(x,y);

dXdt = (2 - Y).*(X - Y);
dYdt = (1+X).* (X + Y);

 un=dXdt./sqrt(dXdt.^2+dYdt.^2);
 wn=dYdt./sqrt(dXdt.^2+dYdt.^2);
 quiver(X,Y,un,wn)
 xlabel('x');
 ylabel('y');
 title('Critical Point at (-2,2)')
 axis tight;
```



Critical Point at (-2,2)
2.5
2.4
2.3
2.2
2.1
y 2
1.9
1.8
1.7
1.6
1.5
-2.5
-2.4
-2.3
-2.2
-2.1
-2
-1.9
-1.8
-1.7
-1.6
-1.5
x

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Simple Competition Model

$$x' = ax - bxy = x(a - by) = F(x, y)$$
$$y' = my - nxy = y(m - nx) = G(x, y)$$
$$a, b, m, n > 0$$

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Search For Equilibrium Points

$x' = 0$ at $x = 0, y = \frac{a}{b}$

$y' = 0$ at $y = 0, x = \frac{m}{n}$

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Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits

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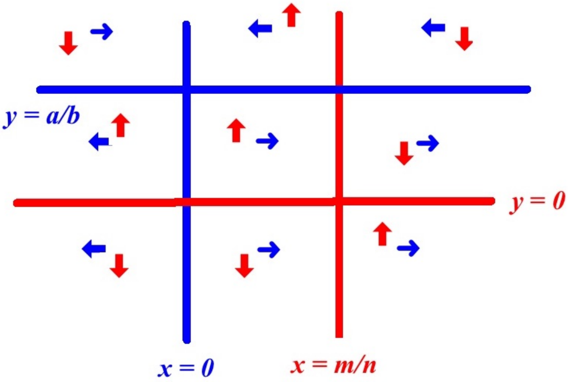
Critical Points: (0,0) and $(\frac{m}{n}, \frac{a}{b})$: One Point Orbits

x-axis is an orbit and y-axis is an orbit.

$x' > 0$	$y' > 0$
$x(a - by) > 0$	$y(m - nx) > 0$
Both Positive	Both Positive
$x > 0$ and $y < \frac{a}{b}$	$y > 0$ and $x < \frac{m}{n}$
OR	OR
Both Negative	Both Negative
$x < 0$ and $y > \frac{a}{b}$	$y >=< 0$ and $x > \frac{m}{n}$

$x' > 0$	$y' > 0$
$x(a - by) > 0$ Both Positive $x > 0$ and $y < \frac{a}{b}$ OR Both Negative $x < 0$ and $y > \frac{a}{b}$	$y(m - nx) > 0$ Both Positive $y > 0$ and $x < \frac{m}{n}$ OR Both Negative $y >=< 0$ and $x > \frac{m}{n}$



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$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$

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$$J(0,0) = \begin{bmatrix} a & 0 \\ 0 & m \end{bmatrix}$$

Both Eigenvalues are positive

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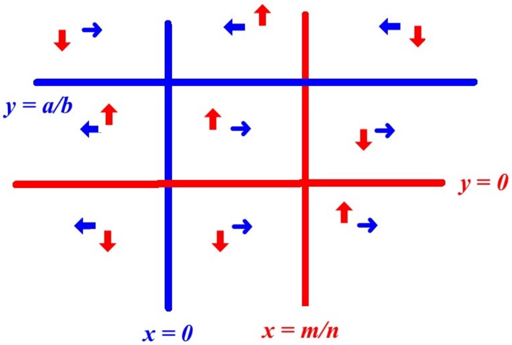

$$F_x = a - by \qquad G_x = -ny$$
$$F_y = -bx \qquad G_y = m - nx$$
$$J\left(\frac{m}{n}, \frac{a}{b}\right) = \begin{bmatrix} a - b\frac{a}{b} & -b\frac{m}{n} \\ -n\frac{a}{b} & m - n\frac{m}{n} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bm}{n} \\ -\frac{na}{b} & 0 \end{bmatrix}$$

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Characteristic Polynomial $\lambda^2 - \frac{-an}{b}\frac{bm}{n} = \lambda^2 - am$

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Eigenvalues $\pm\sqrt{am}$ so we have Saddle Point.

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Eigenvalues $\pm\sqrt{am}$ so we have Saddle Point.

Eigenvectors: $\mathbf{v} = \begin{pmatrix} -bm \\ n\sqrt{am} \end{pmatrix}, \mathbf{w} = \begin{pmatrix} bm \\ n\sqrt{am} \end{pmatrix}$

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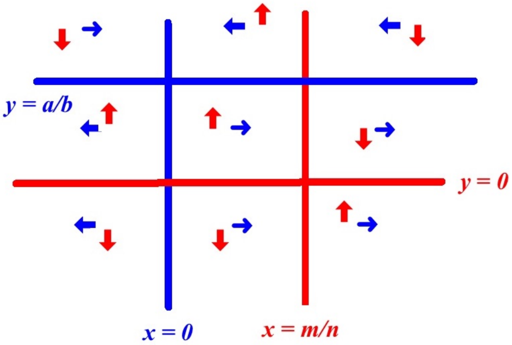

A Specific Example: $a = .9, b = .6, m = .4, n = .3$

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Critical Point $(\frac{.4}{.3}, \frac{.9}{.6}) = (\frac{4}{3}, \frac{3}{2})$

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Critical Point $(\frac{.4}{.3}, \frac{.9}{.6}) = (\frac{4}{3}, \frac{3}{2})$

$$J() = \begin{bmatrix} 0 & -\frac{(.6)(.4)}{.3} \\ -\frac{.27}{.6} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{4}{5} \\ -\frac{9}{20} & 0 \end{bmatrix}$$

A Specific Example: $a = .9, b = .6, m = .4, n = .3$

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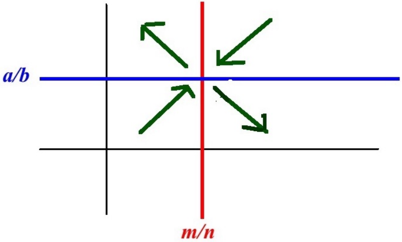
Eigenvalue	Eigenvector	Solution
$\lambda = \sqrt{36/100} = \frac{3}{5}$	$\mathbf{v} = \begin{pmatrix} -4/3 \\ 1 \end{pmatrix}$	$e^{\frac{3}{5}t}\mathbf{v}$
$\lambda = -\sqrt{(36/100} = -\frac{3}{5}$	$\mathbf{w} = \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$	$e^{-\frac{3}{5}t}\mathbf{w}$

General Solution of Linear System

$$C_1 e^{\frac{3}{5}t} \begin{pmatrix} -4/3 \\ 1 \end{pmatrix} + C_2 e^{-\frac{3}{5}t} \begin{pmatrix} 4/3 \\ 1 \end{pmatrix}$$

Behavior Near Equilibrium Point



$$C_1 e^{\frac{3}{5}t}\begin{pmatrix}-4/3\\1\end{pmatrix}+C_2 e^{-\frac{3}{5}t}\begin{pmatrix}4/3\\1\end{pmatrix}$$

Simple Model $x' = ax - bxy$

$y' = my - nxy$

$a, b, m, n > 0$

Each Species Grows **Exponentially** in Absence of Other.

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More Realistic Model

$x' = ax - px^2 - bxy$

$y' = my - qy^2 - nxy$

$a, b, m, n > 0$

Each Spcies Grows **Logistically** in Absence of Other.

Next Time

Predator-Prey Models