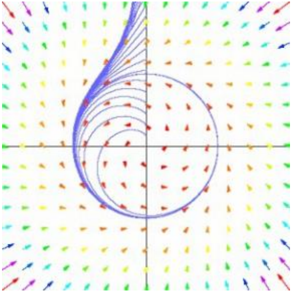
MATH 226: Differential Equations



Class 28: Wednesday, April 23, 2025

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Periodic Solutions and Limit Cycles
Converting From Cartesian To Polar
MATLAB: LimitCycle (Handouts Folder)
Maple: Limit Cycles (Handouts Folder)

Schedule

Today: Converting Between Cartesian and Polar Coordinates

Periodic Solutions and Limit Cycles

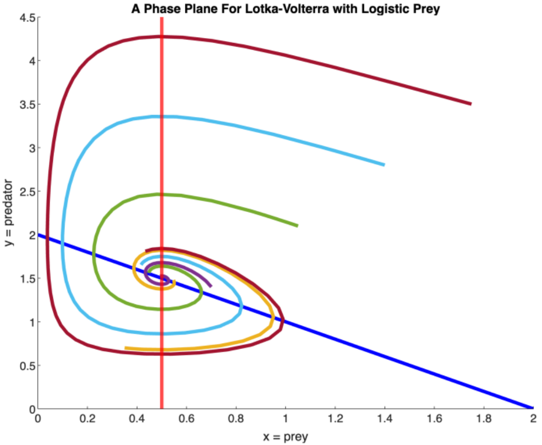
Friday: Chaos and Strange Attractors

Phase Plane for a Lotka-Volterra Predator - Prey with Logistic Prey Growth Model $a/p > m/n$

$$x' = ax - px^2 = bxy$$

$$y' = -my + nxy$$

$$a = 1;\ b = .5;\ p = .5;\ m = .25;\ n = .5;$$


A Phase Plane For Lotka-Volterra with Logistic Prey
y = predator
x = prey
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Phase Plane for a Lotka-Volterra Predator - Prey with Logistic Prey Growth Model *a/p >m/n*

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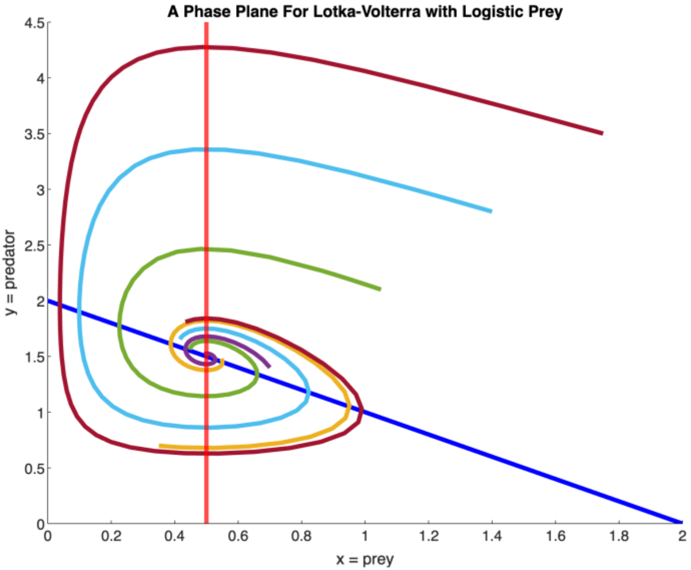
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0
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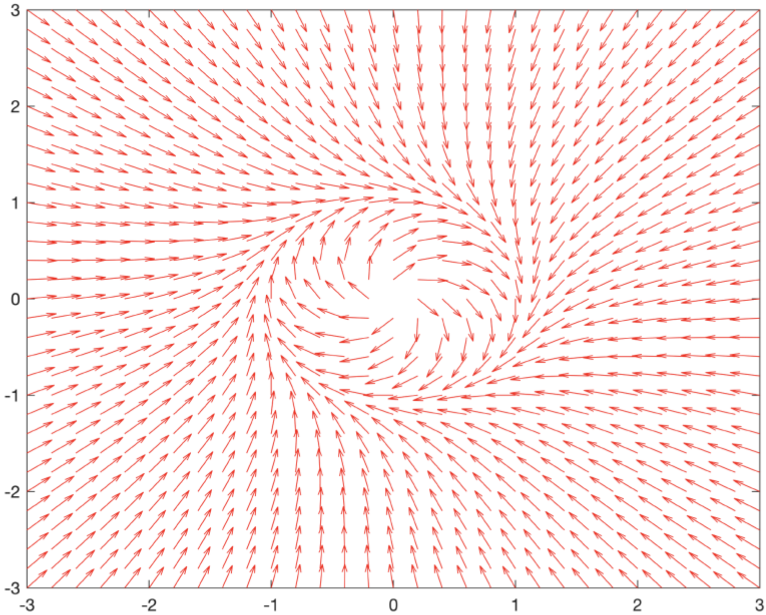
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
x = prey

Limit Cycles

A New Behavior Not Seen in Linear Systems

$$x' = y + x(1 - x^2 - y^2)$$

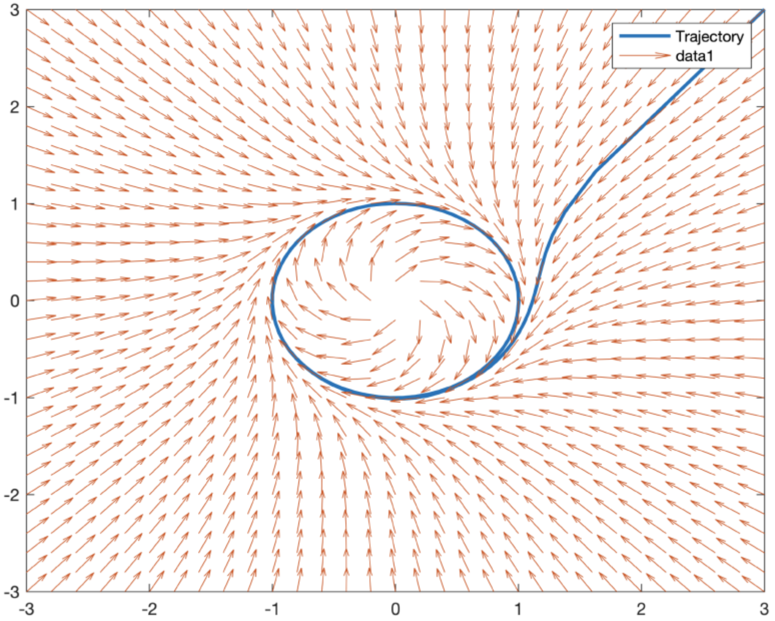
$$y' = -x + y(1 - x^2 - y^2)$$

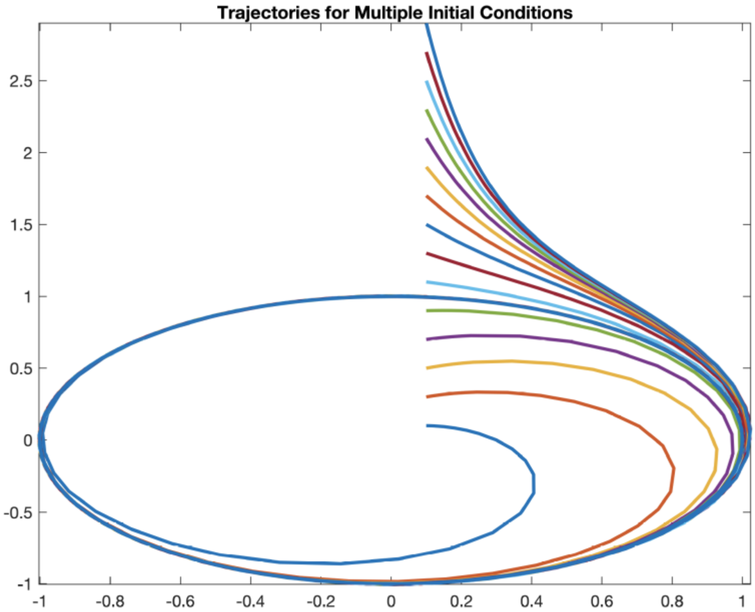


3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3

MATLAB Code

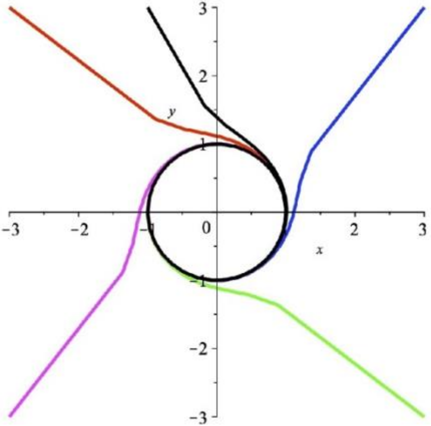
```
[x,y]=meshgrid(-3:.2:3,-3:.2:3);
xprime = y + x .* (1 - x.*x - y.*y);
yprime = -x + y .* (1 - x.*x - y.*y);
L = sqrt(xprime.^2 + yprime.^2);
dyu=yprime./L;
dxu=xprime./L;
quiver(x,y,dxu,dyu, 'r')
```


Trajectory
data1
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3



Trajectories for Multiple Initial Conditions
2.5
2
1.5
1
0.5
0
-0.5
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

```
DEplot({ode1, ode2}, [x(t), y(t)], t = 0..10, x = -3..3, y = -3..3, [[x(0) = 3, y(0) = 3], [x(0) = -3,
  y(0) = 3], [x(0) = 3, y(0) = -3], [x(0) = -3, y(0) = -3], [x(0) = -1, y(0) = 3]], linecolor
  = [blue, red, green, magenta, black], arrows = none, animate = true)
```



Polar Coordinate Version

$ode3 := r'(t) = r(t)\cdot\left(1 - r(t)\right):$

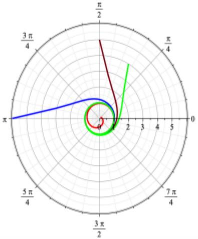
$ode4 := \text{theta}'(t) = -1:$

$P1 := DEplot\left(\{ode3, ode4\}, [r(t), \text{theta}(t)], t = 0..10, \left[\left[\text{theta}(0) = \frac{\text{Pi}}{2}, r(0) = 5\right], [\text{theta}(0) = \text{Pi}, r(0) = 6], \left[\text{theta}(0) = \frac{\text{Pi}}{4}, r(0) = .1\right], \left[\text{theta}(0) = \frac{\text{Pi}}{3}, r(0) = 4\right]\right], arrows = none, linecolor = [burgundy, blue, red, green]\right):$

$conv := plottools{:}\text{-}transform\left((a, b) \rightarrow [a*\cos(b), a*\sin(b)]\right):$

$with(plots):$

$display(conv(P1), axiscoordinates = \text{polar});$



Limit Cycles

$$x' = y + x\,(1 - x\ 2 - y\ 2)$$

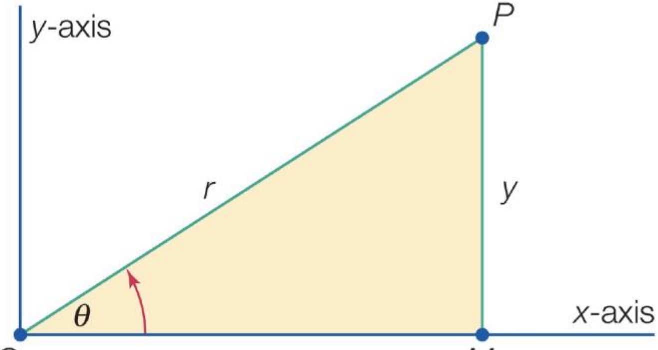
$$y' = -x + y\,(1 - x\ 2 - y\ 2)$$

In Polar Coordinates

$$\boldsymbol{r' = r(1 - r), \theta' = -1}$$

Converting From Cartesian To Polar

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Converting From Cartesian To Polar

$$x = r\cos\theta$$
$$y = r\sin\theta$$

I. Use Product and Chain Rules to Obtain

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta + r(-\sin\theta)\frac{d\theta}{dt}$$
$$\frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r(\cos\theta)\frac{d\theta}{dt}$$

II. $x\,\frac{dx}{dt} + y\frac{dy}{dt}$

$$= (r\cos\theta)\left[\frac{dr}{dt}\cos\theta + r(-\sin\theta)\frac{d\theta}{dt}\right] + (r\sin\ \theta)\left[\frac{dr}{dt}\sin\theta + r(\cos\theta)\frac{d\theta}{dt}\right]$$

$$= r\,\frac{dr}{dt}cos^2\theta - r^2\sin\theta\cos\theta\frac{d\theta}{dt} + r\,\frac{dr}{dt}sin^2\theta + r^2\sin\theta\cos\theta\frac{d\theta}{dt}$$

$$= r\,\frac{dr}{dt}cos^2\theta + r\,\frac{dr}{dt}sin^2\theta = r\,\frac{dr}{dt}[cos^2\theta + sin^2\theta] = r\,\frac{dr}{dt}$$

Thus $\boxed{r\,\frac{dr}{dt} = x\,\frac{dx}{dt} + y\frac{dy}{dt}}$

III. $y\frac{dx}{dt} - x\frac{dy}{dt}$

$$= \left(r\frac{dr}{dt}\sin\theta\cos\theta - r^2 sin^2\theta\frac{d\theta}{dt}\right) - \left(r\frac{dr}{dt}\sin\theta\cos\theta + r^2 cos^2\theta\frac{d\theta}{dt}\right)$$

$$= -r^2(sin^2\theta + cos^2\theta)\frac{d\theta}{dt}$$

$$= -r^2 1\frac{d\theta}{dt}$$

Thus $\boxed{-r^2\frac{d\theta}{dt} = y\frac{dx}{dt} - x\frac{dy}{dt}}$

.

Example: Convert $\begin{cases} \frac{dx}{dt} = y + x - \frac{x}{x^2+y^2} \\ \frac{dy}{dt} = -x + y - \frac{y}{x^2+y^2} \end{cases}$

Solution:

$x\frac{dx}{dt} + y\frac{dy}{dt} = xy + x^2 - \frac{x^2}{x^2+y^2} + -xy + y^2 - \frac{y^2}{x^2+y^2} = x^2 + y^2 - \frac{x^2+y^2}{x^2+y^2}$

$= x^2 + y^2 - 1$

Hence $r\frac{dr}{dt} = r^2 - 1$

and $y\frac{dx}{dt} - x\frac{dy}{dt} = (y^2 + xy - \frac{xy}{x^2+y^2}) - \left(-x^2 + xy - \frac{yx}{x^2+y^2}\right)$

$= y^2 + x^2 = r^2$

So $-r^2\frac{d\theta}{dt} = r^2$ which yields $\frac{d\theta}{dt} = -1$

The converted system looks like $\begin{cases} r\frac{dr}{dt} = r^2 - 1 \\ \frac{d\theta}{dt} = -1 \end{cases}$

Transforming Systems of Differential Equations From Cartesian To Polar Coordinates and Polar To Cartesian

$$\boldsymbol{x\,x' + y\,y' = r\,r'}$$

$$\boldsymbol{y\,x' - x\,y' = -r^2\theta'}$$

Write As:

$$\begin{bmatrix} y & -x \\ x & y \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -r^2\theta' \\ rr' \end{bmatrix}$$

which is equivalent to

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} y & x \\ -x & y \end{bmatrix} \begin{bmatrix} -\theta' \\ r'/r \end{bmatrix}$$

Periodic Solutions and Limit Cycles

Definitions: A **limit cycle** is a closed trajectory in the phase plane such that other nonclosed trajectories spiral toward either from the inside or the outside (or both).

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If the trajectories on one side of the limit cycle spiral toward it while those on the other side move away as $t \to \infty$, then the limit cycle is **semistable**.

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If the trajectories on one side of the limit cycle spiral toward it while those on the other side move away as $t \to \infty$, then the limit cycle is **semistable**.

If the trajectories on both sides of the closed trajectory spiral away as $t \to \infty$, then it is called **unstable**.

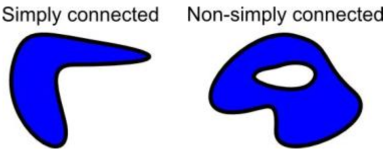
Theorem 1: Let the functions F and G have continuous first partial derivatives in a domain D of the xy-plane. A closed trajectory of the system

$$x' = \frac{dx}{dt} = F(x, y),$$
$$y' = \frac{dy}{dt} = G(x, y)$$

must necessarily enclose at least one critical point.

Moreover, if it encloses only one critical point, that point cannot be a saddle point.

Definition: A two-dimensional domain is **simply connected** if it has no holes; equivalently, any closed loop can be shrunk to a point in the domain.



Theorem 2: Let the functions *F* and *G* have continuous first partial derivatives in a simply connected domain *D* of the *xy*-plane. If $F_x + G_y$ has the same sign throughout *D*, then there is no closed trajectory of the system

$$x' = \frac{dx}{dt} = F(x, y), y' = \frac{dy}{dt} = G(x, y)$$

lying entirely in *D*.

Green's Theorem in the Plane: If C is a sufficiently smooth simple closed curve that is traversed counterclockwise around a region R enclosed by C , then

$$\int_C [F(x,y) - G(x,y)] = \iint_R [F_x(x,y) + G_y(x,y)]dxdy$$

If F and G are continuous functions with continuous first partial derivatives

Theorem 3 (Poincaré-Bendixson Theorem): Let the functions *F* and *G* have continuous first partial derivatives in a domain *D* of the *xy*-plane.

Let D_1 be a bounded subdomain in *D*, and let *R* be the region that consists of D_1 plus its boundary (all points of *R* are in *D*). Suppose that *R* contains no critical points of the system

$$\int_C [F(x,y) - G(x,y)] = \iint_R [F_x(x,y) + G_y(x,y)]dxdy$$

If there exists a constant t_0 such that $x = \varphi(t)$, $y = \psi(t)$ is a solution of the system that exists and stays in R for all $t \geq t_0$, then either

$x = \varphi(t)$, $y = \psi(t)$ is a periodic solution with closed trajectory or

$x = \varphi(t)$, $y = \psi(t)$ has a trajectory that spirals toward a closed trajectory as $t \to \infty$

In either case, the system has a periodic solution in R.

Poincaré – Bendixson Theorem

Henri Poincaré
1854 – 1912
”Sur les courbes définies
une équation différentielle”,
Oeuvres, 1, Paris.
(1892)

Ivar Bendixson
1861 –1935
”Sur les courbes définies par
par des équations différentielles”
Acta Mathematica, Springer Netherlan
24 (1): 1888.