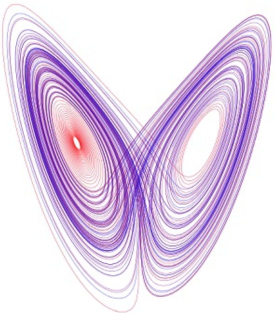
MATH 226 Differential Equations



Class 29: Friday, April 25, 2025

Notes on Assignment 19

Assignment 20

Maple: Chaos

Butterfly Attractor

Team Project 3: Epidemic Models

The Fragility of Being a Center

Consider $\mathbf{X'} = A\mathbf{X}$ with $A = \begin{pmatrix} 36 & 80 \\ -50 & -36 \end{pmatrix}$

Characteristic Polynomial: $\lambda^2 + 2704$ so eigenvalues are $\lambda = \pm 52i$

Suppose we replace 36 with $36 + \epsilon$ where ϵ is a small number.

$$A = \begin{pmatrix} 36+\epsilon & 80 \\ -50 & -36 \end{pmatrix}$$

Characteristic Polynomial: $\lambda^2 - \epsilon\lambda + 2704 - 36\epsilon$

so eigenvalues are

$$\lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 + 144\epsilon - 10816}}{2}$$

ϵ small **positive** means real part $\frac{\epsilon}{2} > 0$: Spiral Source

ϵ small **negative** means real part $\frac{\epsilon}{2} < 0$: Spiral Sink

Chaos and Strange Attractors: The Lorenz Equations

Edward Norton Lorenz

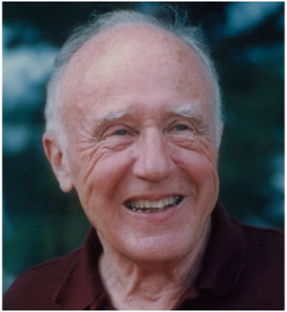
(May 23, 1917 - April 16, 2008)

American mathematician and meteorologist

Pioneer of chaos theory

Discovered the strange attractor concept

Coined the term *Butterfly Effect*.



At one point I decided to repeat some of the computations in order to examine what was happening in greater detail. I stopped the computer, typed in a line of numbers that it had printed out a while earlier, and set it running again. I went down the hall for a cup of coffee and returned after about an hour, during which time the computer had simulated about two months of weather. The numbers being printed were nothing like the old ones. I immediately suspected a weak vacuum tube or some other computer trouble, which was not uncommon, but before calling for service I decided to see just where the mistake had occurred, knowing that this could speed up the servicing process. Instead of a sudden break, I found that the new values at first repeated the old ones, but soon afterward differed by one and then several units in the last decimal place. . . . The numbers I had typed in were not the exact original numbers, but were the rounded off values that had appeared in the original printout. The initial round-off errors were the culprits; they were steadily amplifying until they dominated the solution. In today's terminology, there was chaos.

JOURNAL OF THE ATMOSPHERIC SCIENCES

Deterministic Nonperiodic Flow[1]

EDWARD N. LORENZ

Massachusetts Institute of Technology

(Manuscript received 18 November 1962, in revised form 7 January 1963)

ABSTRACT

Finite systems of deterministic ordinary nonlinear differential equations may be designed to represent forced dissipative hydrodynamic flow. Solutions of these equations can be identified with trajectories in phase space. For those systems with bounded solutions, it is found that nonperiodic solutions are ordinarily unstable with respect to small modifications, so that slightly differing initial states can evolve into considerably different states. Systems with bounded solutions are shown to possess bounded numerical solutions.

A simple system representing cellular convection is solved numerically. All of the solutions are found to be unstable, and almost all of them are nonperiodic.

The feasibility of very-long-range weather prediction is examined in the light of these results.

Lorenz Biography

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, 139th MEETING

Subject..........................Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas?

Author..........................Edward N. Lorenz, Sc.D.
Professor of Meteorology

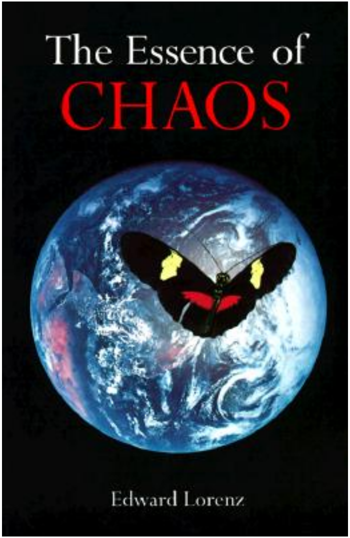
Address..........................Massachusetts Institute of Technology
Cambridge, Mass. 02139

Time..........................10:00 a.m., December 29, 1972

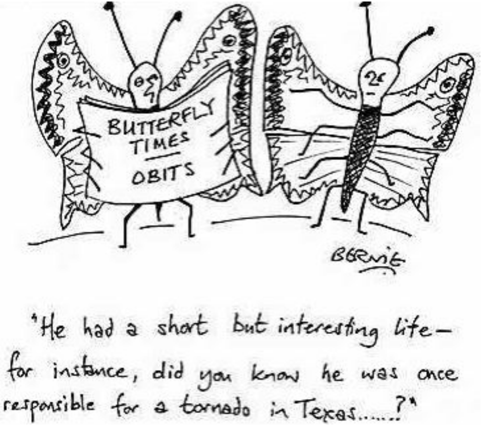
Place..........................Sheraton Park Hotel, Wilmington Room

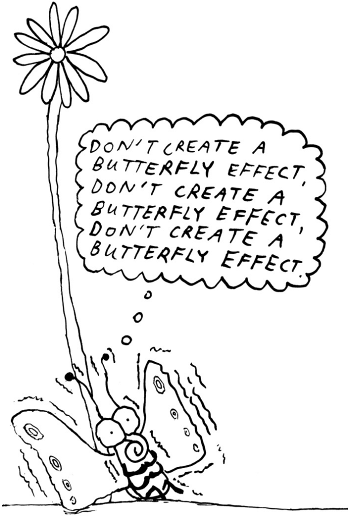
Program..........................AAAS Section on Environmental Sciences
New Approaches to Global Weather: GARP
(The Global Atmospheric Research Program)

Convention Address................Sheraton Park Hotel



Can the Flap of a Butterfly's Wings in Brazil Cause a Tornado in Texas a Week Later?


BUTTERFLY TIMES
OBITS
BERNIE
"He had a short but interesting life — for instance, did you know he was once responsible for a tornado in Texas......?"


DON'T CREATE A BUTTERFLY EFFECT, DON'T CREATE A BUTTERFLY EFFECT, DON'T CREATE A BUTTERFLY EFFECT.

Chaos and Strange Attractors: The Lorenz Equations

$$dx/dt = \sigma(-x + y) = -\sigma x + \sigma y$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

Interesting Values:
$\sigma = 10, b = 8/3, r = 28$

$$dx/dt = -\sigma x + \sigma y = F(x, y, z)$$
$$dy/dt = rx - y - xz = G(x, y, z)$$
$$dz/dt = -bz + xy = H(x, y, z)$$

x: intensity of fluid motion
y, z: Temperature variations in horizontal, vertical
σ, b: material and geometric properties of fluid layer
r is proportional to change in temperature between top and bottom of fluid layer.

The Lorenz equations also arise in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, and chemical reactions.

$$dx/dt = -\sigma x + \sigma y = F(x, y, z)$$
$$dy/dt = rx - y - xz = G(x, y, z)$$
$$dz/dt = -bz + xy = H(x, y, z)$$

Critical Points

$dx/dt = 0$: $y = x$

and $dy/dt = 0$:

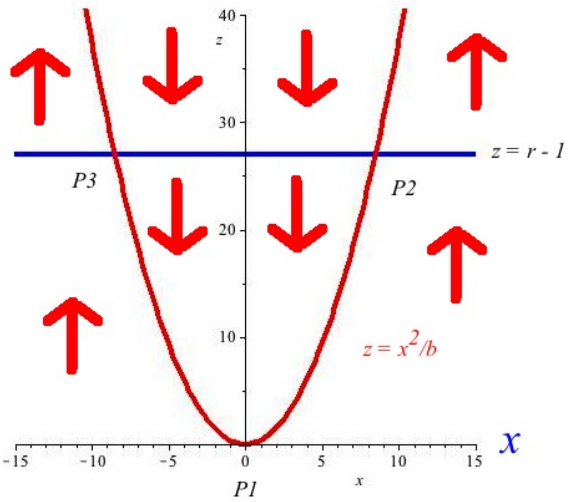
$rx - y - xz =$ implies $rx - x - xz = 0$ so $x(r - 1 - z) = 0$

Thus $x = 0$ or $z = r - 1$

and $dz/dt = 0$ implies $-bz + xy = 0$ so $-bz + x^2 = 0$ or $z = \frac{x^2}{b}$

Note also that $dx/dt > 0$ when $y > x$

and $dz/dt > 0$ when $z < \frac{x^2}{b}$



40
30
20
10
z
z = r - 1
P3
P2
z = x^2/b
x
-15
-10
-5
0
5
10
15
P1
x

Conditions for a Critical Point

$$y = x, z = \frac{x^2}{b}, x = 0 \text{ or } z = r - 1$$

$$P_1 = (x = 0, y = 0, z = 0)$$

$$P_2 = (x = \sqrt{b(r-1)}, y = \sqrt{b(r-1)}, z = r - 1$$

$$P_3 = (x = -\sqrt{b(r-1)}, y = -\sqrt{b(r-1)}, z = r - 1$$

Note: If $r < 1$, then no P_2 or P_3

For $b = 8/3$ and $r = 28$, $P_2 = (6\sqrt{2}, 6\sqrt{2}, 27)$

$$
\begin{aligned}
dx/dt = -\sigma x + \sigma y = F(x,y,z) \\
dy/dt = rx - y - xz = G(x,y,z) \\
dz/dt = -bz + xy = H(x,y,z)
\end{aligned}
$$

Jacobian Matrix

$$
\begin{pmatrix} Fx & Fy & Fz \\ Gx & Gy & Gz \\ Hx & Hy & Hz \end{pmatrix}
=
\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ y & x & -b \end{pmatrix}
$$

Jacobian

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -x \\ y & x & -b \end{pmatrix}$$

At Origin (0,0,0):

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & -0 \\ 0 & 0 & -b \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ r & -1 - \lambda & 0 \\ 0 & 0 & -b - \lambda \end{pmatrix}$$

Expand along third row to find determinant

$$A - \lambda I = \begin{pmatrix} -\sigma - \lambda & \sigma & 0 \\ r & -1 - \lambda & 0 \\ 0 & 0 & -b - \lambda \end{pmatrix}$$

$$det(A - \lambda I) = -(b + \lambda)(\lambda^2 + (1 + \sigma)\lambda + (\sigma - \sigma\lambda)$$

Eigenvalues:

$$\lambda_1 = -b$$

$$\lambda_2 = \frac{-(1+\sigma)+\sqrt{(1+\sigma)^2 - 4\sigma(1-r)}}{2} = \frac{-(1+\sigma)+\sqrt{(1-\sigma)^2+4\sigma r}}{2}$$

$$\lambda_3 = \frac{-(1+\sigma)-\sqrt{(1+\sigma)^2 - 4\sigma(1-r)}}{2} = \frac{-(1+\sigma)-\sqrt{(1-\sigma)^2+4\sigma r}}{2}$$

Note: if σ and r are positive, then all eigenvalues are real and distinct

λ_1 and λ_3 are negative

λ_2 could be positive or negative

For our example, with $\sigma = 10$ and $b = 8/3$, we have

$$\lambda_1 = -8/3$$

$$\lambda_2 = \frac{-11+\sqrt{81+40r}}{2}$$

$$\lambda_3 = \frac{-11-\sqrt{81+40r}}{2}$$

$$\lambda_2 = \frac{-11+\sqrt{81+40r}}{2}$$

λ_2 will be positive if and only if $81 + 40r > 121$; that is, $r > 1$

Thus, origin is asymptotically stable if $r < 1$ and unstable if $r > 1$

In general, examine sign of λ_2

$$-(1+\sigma)+\sqrt{(1-\sigma)^2+4\sigma r}$$

which is positive if

$$\sqrt{(1-\sigma)^2+4\sigma r} > (1+\sigma)$$

Squaring: $(1-\sigma)^2+4\sigma r > (1+\sigma)^2$

$$1-2\sigma+\sigma^2+4\sigma r > 1+2\sigma+\sigma^2$$

$$4\sigma r > 4\sigma$$

$$r > 1$$

So $r = 1$ is a critical value for the origin.

Critical Points

$P_1 = (0,0,0)$ and if $r > 1$:

$P_2 = (\sqrt{b(r-1)},\ \sqrt{b(r-1)},\ r-1)$

$P_3 = (-\sqrt{b(r-1)},\ -\sqrt{b(r-1)},\ r-1)$

If $r \leq 1$, then origin is only critical point.

Suppose $r > 1$ so we have other critical points

Recall

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$$

With $x = y = \sqrt{b(r-1)}, z = r-1$, we have

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

At $x = y = \sqrt{b(r-1)}, z = r - 1$, we have

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$$

Let $\sigma = 10$ and $b = 8/3$

The characteristic polynomial is

$$p(\lambda) = \frac{3\lambda^3 + 41\lambda^2 + 8(r+10)\lambda + 160(r-1)}{3}$$

$0 < r < 1$	$P_1 = (0, 0, 0)$ is unique critical point; asymptotically stable
$1 < r < 1.3456$	$p(\lambda)$ has 3 negative roots P_2, P_3 asymptotically stable; P_1 unstable
$1.3456 < r < 24.737$	$p(\lambda)$ has 1 negative root; P_1 unstable P_2, P_3 asymptotically stable (spiral in);
$24.737 < r$	1 negative root; P_1, P_2, P_3 unstable; Most orbits near P_1, P_2 spiral away

The characteristic polynomial is
$p(\lambda) = \frac{3\lambda^3+41\lambda^2+8(r+10)\lambda+160(r-1)}{3}$

so the sum of the eigenvalues = -4/3.

When $\lambda = -41/3$ is an eigenvalue, real parts of the others are 0.

Roots: $\lambda^*, a + bi$ and $a - bi$; sum of roots is $\lambda^* + 2a$.

For $\lambda > -41/3, a < 0$ and for $\lambda < -41/3, a > 0$

Find r when $p(-41/3) = 0$:

$p(-41/3) = \frac{-3760+152r}{3}$ so $r = \frac{470}{19} = 24.737$