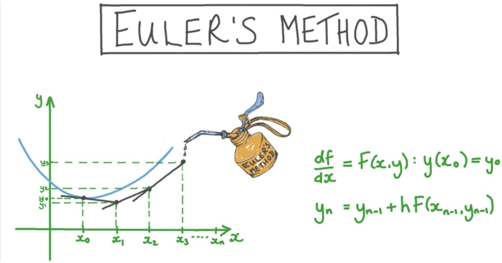
MATH 226 Differential Equations



Class 30: Monday, April 28, 2025

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Handouts

Notes on Assignment 19

Assignment 20

Project 3

Due: Friday, May 9

Numerical Methods
For Studying Differential Equations

Euler's Method

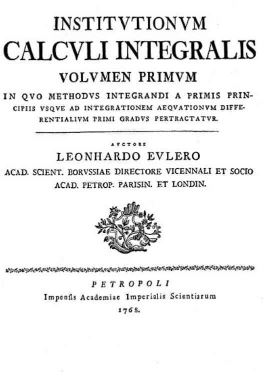
Numerical Accuracy

Improved Euler

Ringe-Kutta Method

Numerical Methods for First Order Systems

Euler's Method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CALCVLI INTEGRALIS
VOLVMEN PRIMVM
IN QVO METHODVS INTEGRANDI A PRIMIS PRINCIPIIS VSQVE AD INTEGRATIONEM AEQVATIONVM DIFFERENTIALIVM PRIMI GRADVS PERTRACTATVR.

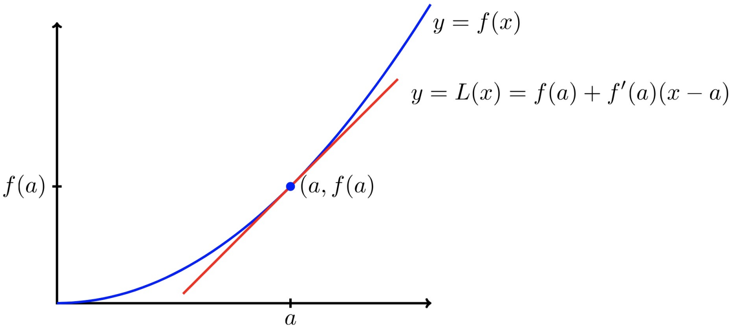
AVCTORE
LEONHARDO EVLERO
ACAD. SCIENT. BORVSSIAE DIRECTORE VICENNALI ET SOCIO
ACAD. PETROP. PARISIN. ET LONDIN.

PETROPOLI
Impenſis Academiae Imperialis Scientiarum
1768.

Leonhard Euler
April 15, 1707 – September 18, 1783
Swiss mathematician, physicist, astronomer, geographer, logician and engineer
Link To Euler's Biography

The Most Important Diagram in Calculus

Linear Approximation

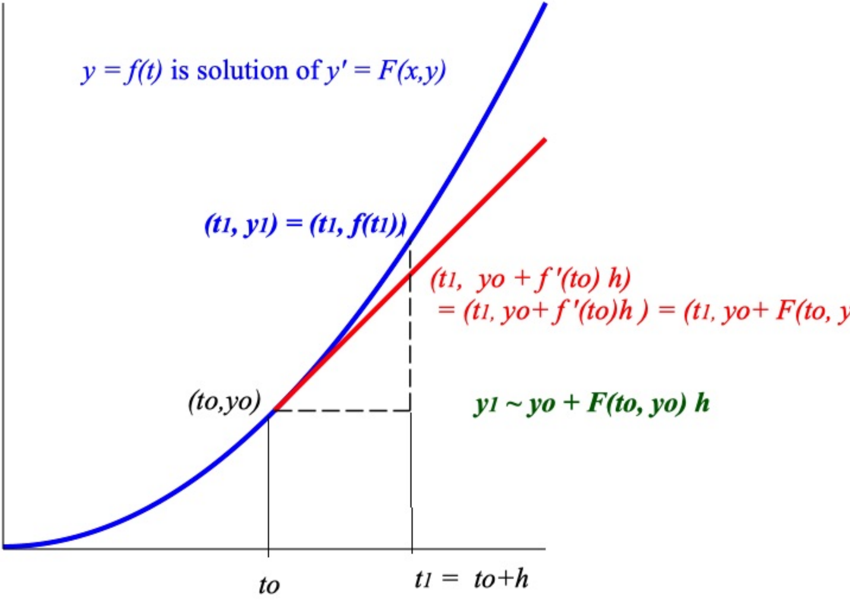


$f(x) \approx L(x)$ for x near a

$$f(x) \approx f(a) + f'(a)h \text{ where } h = x - a$$

Suppose f is the solution of the Differential Equation

$$y' = \frac{dy}{dx} = F(x, y) \text{with initial condition } y_0 = f(x_0)$$


y = f(t) is solution of y′ = F(x,y)
(t1, y1) = (t1, f(t1))
(t1, yo + f ′(to) h)
= (t1, yo+ f ′(to)h) = (t1, yo+ F(to,
y1 ~ yo + F(to, yo) h
(to,yo)
to
t1 = to+h

Given $y' = F(t, y)$ with $f(t_0) = y_0$

$$y_1 = y_0 + F(t_0, y_0)h$$
$$y_2 = y_1 + F(t_1, y_1)h \text{ where } t_1 = t_0 + h$$
$$y_3 = y_2 + F(t_2, y_2)h \text{ where } t_2 = t_1 + h$$
$$y_4 = y_3 + F(t_3, y_3)h \text{ where } t_3 = t_2 + h$$
$$\ldots$$
$$y_{n+1} = y_n + F(t_n, y_n)h \text{ where } t_n = t_{n-1} + h$$

Example: Consider the Differential Equation

$$y' = 3 + t - y \text{ with } y(0) = 1$$

This is First Order Linear:

$$y' + y = 3 + t$$

Integrating Factor is $e^{\int 1dt} = e^t$

$$e^t\, y' + e^t\, y = 3e^t + te^t$$

$$(e^t\, y)' = 3e^t + te^t$$

$$e^t\, y = 3e^t + te^t - e^t + C$$

$$y = 3 + t - 1 + Ce^{-t} = t + 2 + Ce^{-t}$$

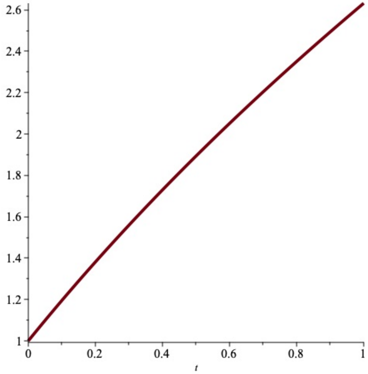
Apply Initial Condition:

$$1 = 0 + 2 + C \text{ so } C = -1$$

Solution is $y = t + 2 - e^{-t}$

$$y' = 3 + t - y \text{ with } y(0) = 1$$

Solution is $y = t + 2 - e^{-t}$



Euler's Method

$$y' = 3 + t - y \text{ with } y(0) = 1$$

Solution is $y = t + 2 - e^{-t}$

Set Step Size $h = 0.1$

$$y_0 = 1$$
$$y_1 = y_0 + y'(0)h = 1 + (3 + 0 - 1)h = 1 + 2h = 1 + 2(.1) = 1.2$$

True Value is $.1 + 2 - e^{-.1} = 1.195162582$

$$y_2 = y_1 + y'(.1)h = 1.2 + (3 + .1 - 1.2)(.1) = 1.2 + .19 = 1.39$$

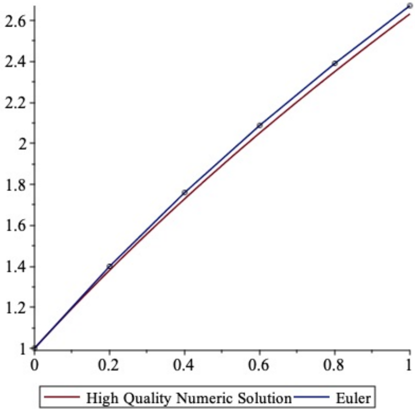
True Value is $.2 + 2 - e^{-.2} = 1.381269247$

t	Exact	$h = 0.1$	$h = 0.05$	$h = 0.025$
0	1.	1.0	1.0	1.0
0.1	1.195162582	1.2	1.1975	1.196312109
0.2	1.381269247	1.39	1.38549375.	1.383348196
0.3	1.559181779	1.571	1.564908109	1.562001654
0.4	1.729679954	1.7439	1.736579569	1.733079832
0.5	1.893469340	1.9095	1.901263061	1.897312320

Euler's Method with $h = 0.1$

$$y' = 3 + t - y \text{ with } y(0) = 1$$

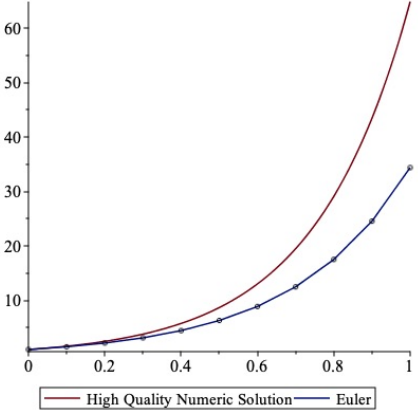
Solution is $y = t + 2 - e^{-t}$



What Can Go Wrong?
Euler's Method with $h = 0.1$

$$y' = 1 - t + 4y \text{ with } y(0) = 1$$

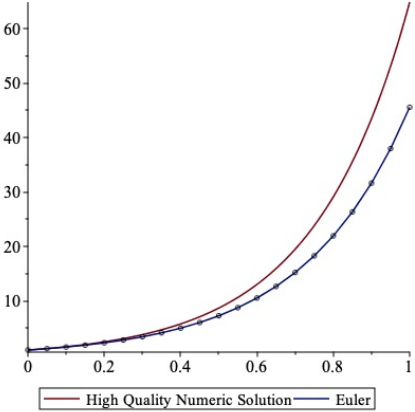
$$\text{Solution is } y = \frac{t}{4} - \frac{3}{16} + \frac{19}{16}e^{4t}$$



What Can Go Wrong?
Cut Step Size in Half
Euler's Method with $h = 0.05$

$$y' = 1 - t + 4y \text{ with } y(0) = 1$$

$$\text{Solution is } y = \frac{t}{4} - \frac{3}{16} + \frac{19}{16}e^{4t}$$



What Can Go Wrong?
Euler's Method with $h = 0.05$ but extend the interval to [0,2]

$$y' = 1 - t + 4y \text{ with } y(0) = 1$$

$$\text{Solution is } y = \frac{t}{4} - \frac{3}{16} + \frac{19}{16}e^{4t}$$