

MATH 226 Differential Equations



Class 31: April 30, 2025



Heun's Method in MATLAB

Announcements
Project 3 Due
Next Friday

Euler's Method

Given $y' = f(t, y)$ with $y(t_0) = y_0$

Solution: Function $\phi(t)$ so $\phi'(t) = f(t, \phi(t))$, $\phi(t_0) = y_0$

For a sequence of times $t_0, t_1, t_2, t_3, \dots, t_n, t_{n+1}, \dots$,
we want to estimate

$$\phi(t_0), \phi(t_1), \phi(t_2), \dots, \phi(t_n), \phi(t_{n+1}), \dots$$

Euler: Use $y_0, y_1, y_2, \dots, y_n, y_{n+1}, \dots$ where

$$y_0 = \phi(t_0)$$

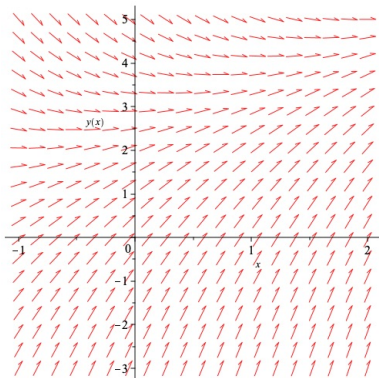
$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0) = y_0 + f(t_0, y_0)h$$

$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1) = y_1 + f(t_1, y_1)h$$

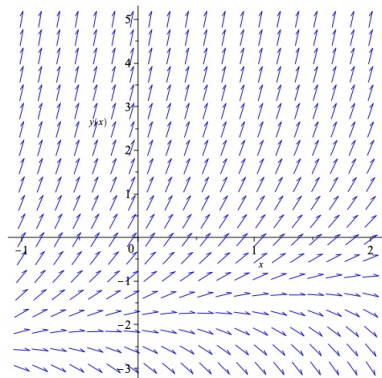
$\dots$

$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n) = y_n + f(t_n, y_n)h$$

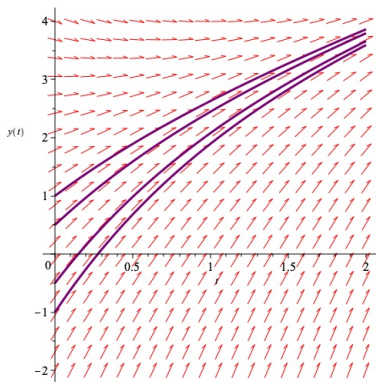
$\dots$ (if each $t_{i+1} - t_i = h$)



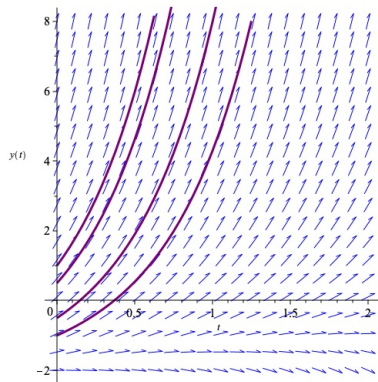
$$y' = 3 + t - y$$



$$y' = 4 - t + 2 * t$$



$$y' = 3 + t - y$$



$$y' = 4 - t + 2 * t$$

Another Way To Look at Euler's Method

The Differential Equation asserts that at each t_n , we have

$$\phi'(t_n) = f(t_n, \phi(t_n))$$

Approximate the Derivative on left by Difference Quotient:

$$\frac{\phi(t_{n+1}) - \phi(t_n)}{t_{n+1} - t_n} \approx f(t_n, \phi(t_n))$$

Using approximate values y_{n+1} and y_n , we have

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} \approx f(t_n, \phi(t_n))$$

which is Euler's formula.

Yet Another Approach

Start with the Differential Equation

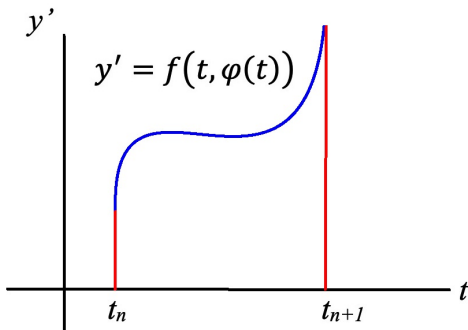
$$\phi'(t) = f(t, \phi(t))$$

Integrate both sides over the interval $[t_n, t_{n+1}]$:

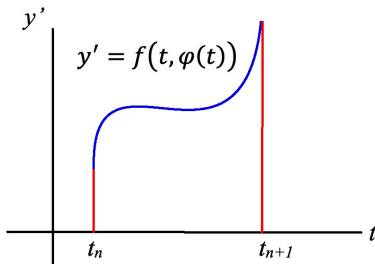
$$\int_{t_n}^{t_{n+1}} \phi'(t) dt = \int_{t_n}^{t_{n+1}} f(t, \phi(t)) dt$$

Now left hand side is $\phi(t_{n+1}) - \phi(t_n)$

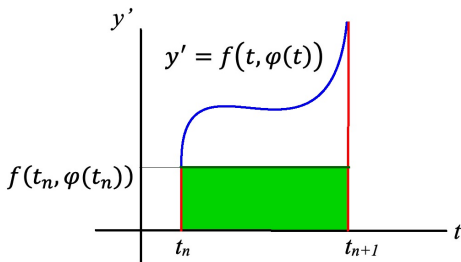
and right hand side is area under curve



Thus $\phi(t_{n+1}) - \phi(t_n) = \text{Area Under Curve}$

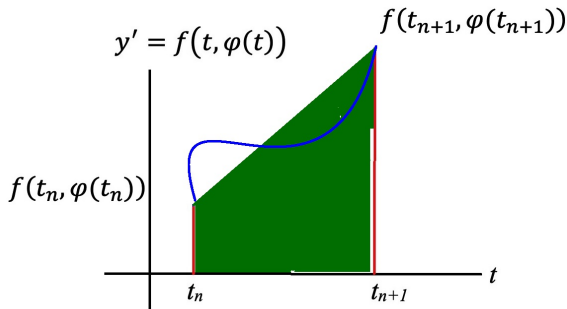


Approximate by Area of Green Rectangle $(f(t_n, \phi(t_n)))(t_{n+1} - t_n)$



$$\phi(t_{n+1}) - \phi(t_n) = \text{Area Under Curve}$$

A Better Way to Estimate Area Under the Curve



$$\text{Trapezoid Rule: } A = \frac{1}{2} [f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1}))] (t_{n+1} - t_n)$$

$$\text{Thus } \phi(t_{n+1}) - \phi(t_n) \approx \frac{1}{2} [f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1}))] (t_{n+1} - t_n)$$

$$\phi(t_{n+1}) - \phi(t_n) \approx \frac{1}{2} [f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1}))] (t_{n+1} - t_n)$$

Replace $\phi(t)$'s with y 's:

$$y_{n+1} - y_n \approx \frac{1}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})] (t_{n+1} - t_n)$$

$$y_{n+1} \approx y_n + \frac{1}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1})] (t_{n+1} - t_n)$$

But we don't know y_{n+1} so we can't find $f(t_{n+1}, y_{n+1})$

Solution: Use Euler Estimate: $y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$

Setting $h = t_{n+1} - t_n$, we have

$$y_{n+1} = y_n + \frac{1}{2} [f(t_n, y_n) + f(t_{n+1}, y_n + f(t_n, y_n)h)] h$$

$$y_{n+1} = y_n + \frac{1}{2} [f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)] h$$

$$y_{n+1} = y_n + \frac{1}{2} [f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)] h$$

$$y_{n+1} = y_n + \frac{f(t_n, y_n) + f(t_n + h, y_n + f(t_n, y_n)h)}{2} h$$

If we let $f_n = f(t_n, y_n)$, then this formula becomes

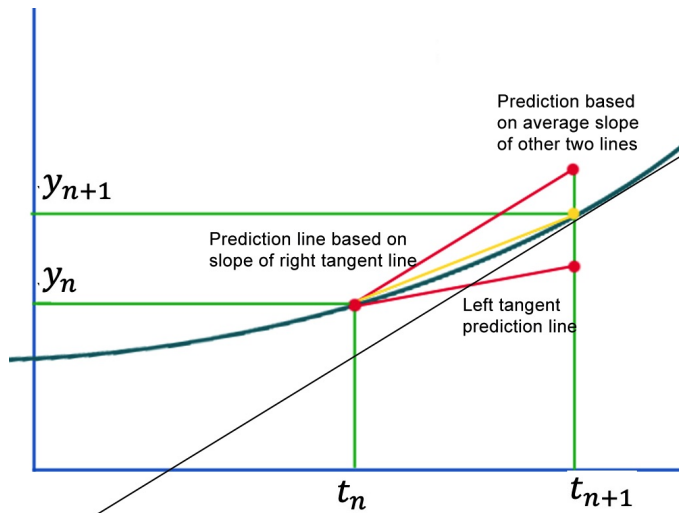
$$y_{n+1} = y_n + \frac{f_n + f(t_n + h, y_n + f_n h)}{2} h$$

Improved Euler's Method or Heun's Method



Karl Heun

April 3, 1859 – January 10, 1929



Heun's Method

Step 1 (Predictor): Calculate intermediate value y_{n+1}^*

$$y_{n+1}^* = y_n + hf(t_n, y_n)$$

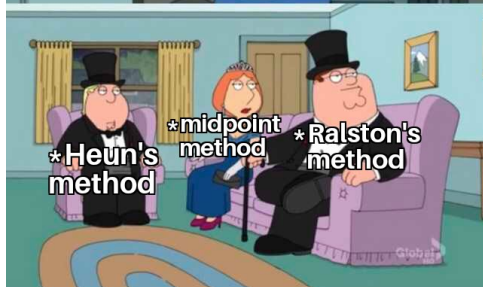
Step 2 (Corrector): Calculate final approximation y_{n+1}

$$y_{n+1} = y_n + \frac{h}{2} [f(t_n, y_n) + f(t_{n+1}, y_{n+1}^*)]$$

We use Euler's Method to estimate roughly the coordinates of the next point in the solution, and with this knowledge, the original estimate is re-predicted or corrected.

Assuming that the quantity $f(t, y)$ on the right hand side of the equation can be thought of as the slope of the solution sought at any point (t, y) , we combine this with the Euler estimate of the next point to give the slope of the tangent line at the right end-point.

Next the average of both slopes is used to find the corrected coordinates of the right end interval.



Second Order Methods

$$y_{n+1} = y_n + h(ak_1 + bk_2) \text{ where}$$

$$k_1 = f(t_n, y_n)$$

$$k_2 = f(t_n + ph, y_n + qk_1h)$$

$$\text{with } a + b = 1, bp = bq = \frac{1}{2}$$

Heun	$a = b = 1/2$	$p = q = 1$
Midpoint	$a = 0, b = 1$	$p = q = 1/2$
Ralston	$a = 1/4, b = 3/4$	$p = q = 2/3$

Anthony Ralston, "Runge-Kutta Methods with Minimum Error Bounds" (1962)

[Ralston Paper](#)



Anthony Ralston
1930 –