MATH 226 Differential Equations

Class 32: Friday, May 2, 2025

Notes on Assignment 20
Assignment 21

Announcements

Project 3 Due Next Friday

Final Exam
Friday, May 16: 9 AM - Noon

Numerical Approximations To Solutions of Differential Equations

$$y' = f(t, y) \text{ with } y(t_0) = y_0$$

Euler's Method: $y_{n+1} = y_n + f(t_n, y_n)h$

Improved Euler's:
$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n)_+ f(t_n + h, y_n + f(t_n, y_n))\right]$$

Second Order Methods

$$y_{n+1} = y_n + h\left(ak_1 + bk_2\right) \text{ where}$$

$$k_1 = f(t_n, y_n)$$
$$k_2 = f(t_n + ph, y_n + qk_1 h)$$

with $a + b = 1, bp = bq = \frac{1}{2}$

Heun	$a = b = 1/2$	$p = q = 1$
Midpoint	$a = 0, b = 1$	$p = q = 1/2$
Ralston	$a = 1/4, b = 3/4$	$p = q = 2/3$

Start with the Differential Equation

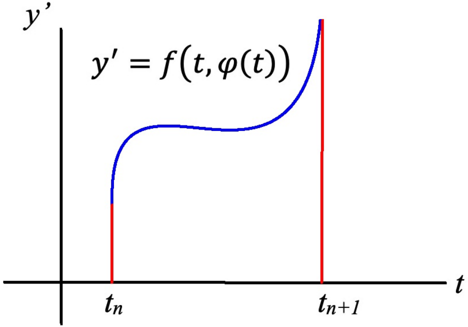
Solution ϕ has $\phi'(t) = f(t_n, \phi(t))$

Integrate both sides over the interval $[t_n, t_{n+1}]$:

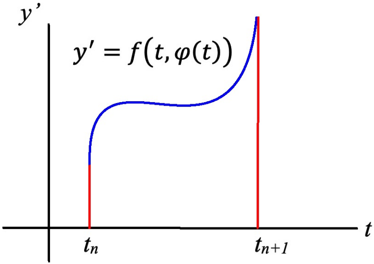
$$\int_{t_n}^{t_{n+1}} \phi'(t)\, dt = \int_{t_n}^{t_{n+1}} f(t_n, \phi(t))\, dt$$

Now left hand side is $\phi(t_{n+1}) - \phi(t_n)$

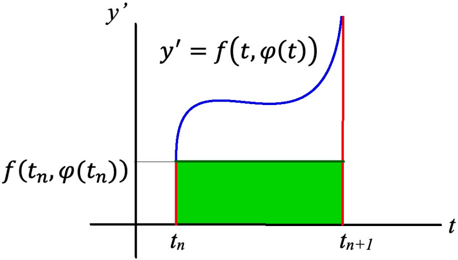
and right hand side is area under curve



Thus $\phi(t_{n+1}) - \phi(t_n) =$ Area Under Curve

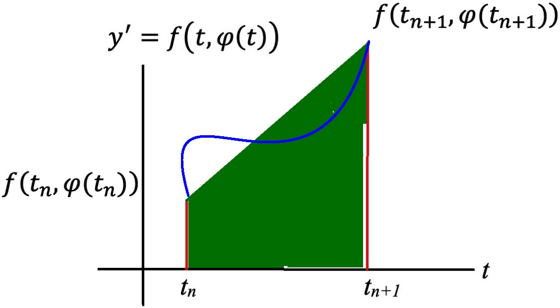


Approximate by Area of Green Rectangle $(f(t_n, \phi(t_n))(t_{n+1} - t_n)$



$$\phi(t_{n+1}) - \phi(t_n) = \text{Area Under Curve}$$

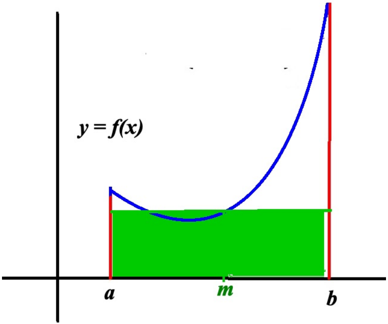
A Better Way to Estimate Area Under the Curve



$$\text{Trapezoid Rule:} A = \frac{1}{2}\left[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1})\right](t_{n+1} - t_n)$$

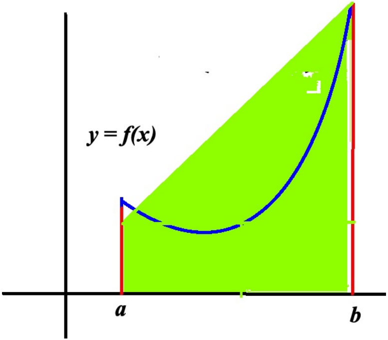
$$\text{Thus } \phi(t_{n+1}) - \phi(t_n) \approx \frac{1}{2}\left[f(t_n, \phi(t_n)) + f(t_{n+1}, \phi(t_{n+1})\right](t_{n+1} - t_n)$$

Other Ways To Estimate Area Under Curve



Midpoint Rule

$(b-a)f(m)$



Trapezoid Rule

$\frac{b-a}{2}[f(a)+f(b)]$

Midpoint Rule: $(b-a)f(m)$

Trapezoidal Rule: $\dfrac{b-a}{2}[f(a)+f(b)]$

Weighted Average of Midpoint Rule and Trapezoidal Rule

Give Weight 2/3 to Midpoint and 1/3 to Trapezoid

$$(b-a)\left[\frac{2}{3}f(m)+\frac{1}{3}\frac{f(a)+f(b)}{2}\right]=\frac{b-a}{3}\left[\frac{4f(m)+f(a)+f(b)}{2}\right]$$

$$\frac{b-a}{6}[f(a)+4f(m)+f(b)]=\frac{h}{3}[f(a)+4f(m)+f(b)]$$

$$h=\frac{b-a}{2} \text{ and } m=\frac{a+b}{2}$$

Thomas Simpson
August 20, 1710 – May 14,1761
Simpson's Rule

Johannes Kepler
Dec. 27, 1571 – Nov. 15, 1630
Keplersche Fassregel

Approximating f on $[a,b]$ with parabola

Given $f(a) = L, f\left(\frac{a+b}{2}\right) M, f(b) = R$

With $y = Ax^2 + Bx + C$

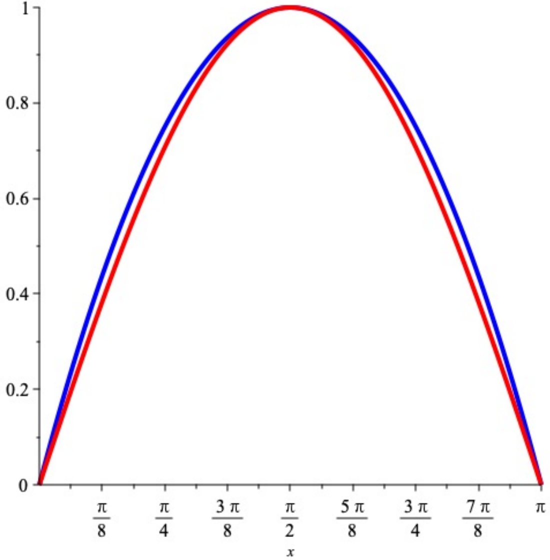
$$a^2A + aB + 1C = L$$
$$\left(\frac{a+b}{2}\right)^2 A + \left(\frac{a+b}{2}\right) B + 1C = M$$
$$b^2A + bB + 1C = R$$

$$\begin{pmatrix} a^2 & a & 1 \\ \left(\frac{a+b}{2}\right)^2 & \left(\frac{a+b}{2}\right) & 1 \\ b^2 & b & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} L \\ M \\ R \end{pmatrix}$$

With $a = 0$ and $b = 1$, we get

A = 2L - 4M + 2R B = -3L - R + 4M C = L

Example: Parabola Approximation of $\sin x$ on $[0, \pi]$



$\sin x$

$-\frac{4}{\pi^2}x^2 + \frac{4}{\pi}x$

Approximate function by the quadratic polynomial (i.e. parabola) $P(x)$ that takes the same values as the function at the end points a and b and the midpoint $m = (a + b)/2$.

We obtain $P(x) =$

$$f(a)\frac{(x-m)(x-b)}{(a-m)(a-b)}+f(m)\frac{(x-a)(x-b)}{(m-a)(m-b)}+f(b)\frac{(x-m)(x-a)}{(b-a)(b-m)}$$

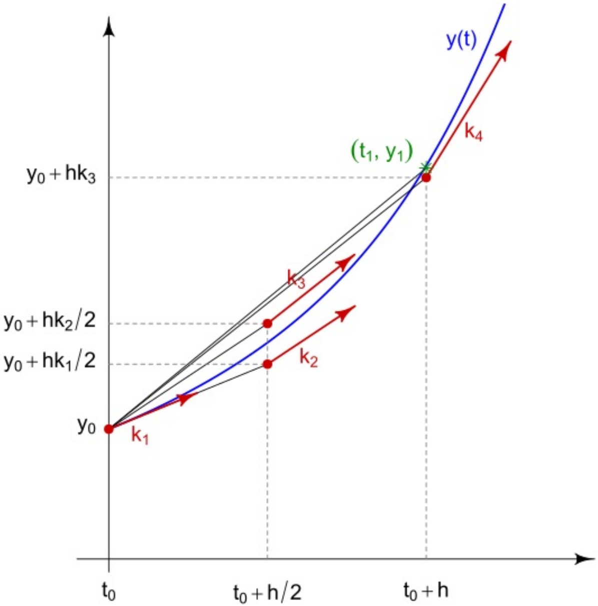
Then $\displaystyle\int_a^b P(x) = \frac{b-a}{6}\left[f(a) + 4f(m) + f(b)\right]$

The Classic Runge–Kutta Method (RK4)

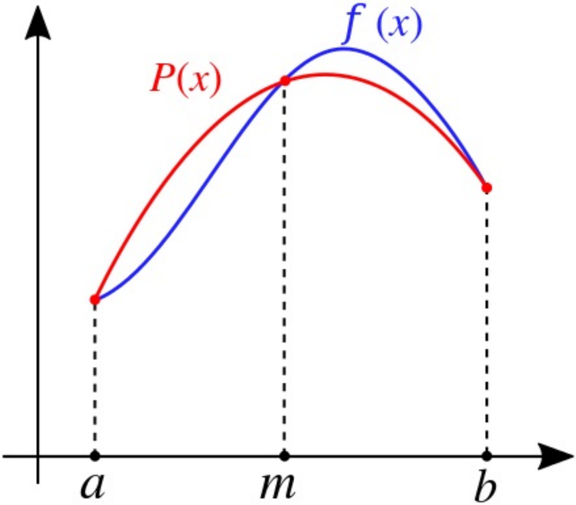
$$y_{n+1} = y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

where

$$\begin{aligned}
k_1 &= f(t_n,\ y_n) \\
k_2 &= f(t_n + \frac{h}{2},\ y_n + \frac{k_1}{2}) \\
k_3 &= f(t_n + \frac{h}{2},\ y_n + h\frac{k_2}{2}) \\
k_4 &= f(t_n + h,\ y_n + hk_3)
\end{aligned}$$


y(t)
k_4
(t_1, y_1)
$y_0 + hk_3$
k_3
$y_0 + hk_2/2$
$y_0 + hk_1/2$
k_2
y_0
k_1
t_0
$t_0 + h/2$
$t_0 + h$

Motivation: Simpson's Rule



$$\int_a^b f(x)\,dx \approx \frac{h}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

More General Runge–Kutta Methods

$$y_{n+1} = y_n + h \sum_{i=1}^{s} b_i k_i,$$

where

$$\begin{aligned}
k_1 &= f(t_n, y_n), \\
k_2 &= f(t_n + c_2 h, y_n + h(a_{21} k_1)), \\
k_3 &= f(t_n + c_3 h, y_n + h(a_{31} k_1 + a_{32} k_2)), \\
&\vdots \\
k_s &= f(t_n + c_s h, y_n + h(a_{s1} k_1 + a_{s2} k_2 + \cdots + a_{s,s-1} k_{s-1})).
\end{aligned}$$

Carl David Tolmé Runge
August 30, 1856
– January 3, 1927
Runge Biography

Martin Wilhelm Kutta
November 3, 1867
– December 25, 1944
Kutta Biography

Error Estimates Proportional to Step Size h

Method	Local Error	Global Error
Euler	h^2	h
Improved Euler	h^3	h^2
Runge–Kutta	h^5	h^4

Example: $y' = 1 - t + 4y$ with $y(0) = 1$

Predict Value of y at $t = 2$

Improved Euler	Runge–Kutta	Runge–Kutta	Runge–Kutta	Exact
$h = .0025$	$h = 0.2$	$h = 0.1$	$h = .05$	
$h = 1/40$	$h = 1/5$	$h = 1/10$	$h = 1/20$	
3496.6702	3490.5574	3535.8667	3539.8804	3540.2

Comparing Improved Euler and Runge–Kutta

160 Functional Evaluations

Improved Euler ($h = .025$) 1.23%. Error

Runge–Kutta ($h = .05$) .00903% Error

Method	h	Evaluations	Percent Error
Improved Euler	.025	160	1.23%
Runge-Kutta	.2	40	1.4%

Runge–Kutta produces

Better Results With Similar Effort

Similar Results with Less Effort

Taylor Series Approach

$$y(t) = y(t_n) + y'(t_n)(t - t_n) + \frac{y''(t_n)}{2!}(t - t_n)^2 + \frac{y^{(3)}(t_n)}{3!}(t - t_n)^3 + \ldots$$