# MATH 226 Differential Equations

Class 34: Wednesday, May 9, 2025

Handouts

# Assignment 23

# Announcements

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## Project 3 Due Friday

**Course Response Forms**
**In Class Next Monday**
**Bring Laptop/SmartPhone**

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**Final Exam**
**Friday, May 16: 9 AM - Noon**

## Ways To Study Differential Equations

- Numerical Approximations to Solution
- Linearize Near Equilibrium Points
- Qualitative Analysis

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- Linearize Near Equilibrium Points
- Qualitative Analysis
- **SOLVE THE EQUATION**

# Example

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Integrate Both Sides With Respect to $t$

$$\ln y = t + C$$

Exponentiate

$$y = Ce^t$$

Use Initial Condition: $y = 1$ when $t = 0$

$$1 = Ce^0 = C \times 1 = C$$

$y = e^t$ is the unique solution

$$e^t$$

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**What Is Going On Inside The Calculator?**

## Power Series Approach

Assume solution can be represented by a power series

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Then we have an expression for the solution if we can determine what the constant coeffients $a_0, a_1, a_2, ..$ are. Example: Solve

$$y' = y \text{ with initial value } y(0) = 1$$

With

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .a_4 x^4 + a_5 x^5...$$

we have

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ...$$

Since $y' = y$ and

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + .a_4x^4 + a_5x^5...$$
$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + ...$$

we can equate coefficients

$$a_1 = a_0 \quad 2a_2 = a_1 \quad 3a_3 = a_2 \quad . \; . \; . \quad na_n = a_{n-1}$$

Since $y' = y$ and

$$\begin{aligned} y =& a_0 + a_1 x + a_2 x^2 + a_3 x^3 + .a_4 x^4 + a_5 x^5 ... \\ y' =& a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + ... \end{aligned}$$

we can equate coefficients

$$\begin{array}{lllll} a_1 = a_0 & 2a_2 = a_1 & 3a_3 = a_2 & \ldots & na_n = a_{n-1} \\ a_1 = a_0 & a_2 = \frac{a_1}{2} & a_3 = \frac{a_2}{3} & \ldots & a_n = \frac{a_{n-1}}{n} \end{array}$$

Since $y' = y$ and

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$$a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$
$$a_3 = \frac{a_2}{3} = \frac{a_0}{3 \times 2} = \frac{a_0}{3!}$$
$$a_4 = \frac{a_3}{4} = \frac{a_0}{4 \times 3!} = \frac{a_0}{4!}$$
$$\ldots$$
$$a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n \times (n-1)!} = \frac{a_0}{n!}$$

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$$...$$
$$a_n = \frac{a_{n-1}}{n} = \frac{a_0}{n\times(n-1)!} = \frac{a_0}{n!}$$

So the solution of $y' = y$ is

$$y = a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...\right)$$

Solution to $y' = y$ with $y(0) = 1$ has the form

$$y = a_0 \left( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ... \right)$$

Evaluating at $x = 0$ gives $1 = y(0) = a_0(1 + 0 + 0 + 0 + ...) = a_0$ so $a_0 = 1$ and

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But we also know the solution is $y = e^x$. Thus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^n}{n!} + ...$$

$$P_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$$

| $n$ | $P_n(1)$ |
|---|---|
| 0 | 1.0 |
| 1 | 2.0 |
| 2 | 2.50000 |
| 3 | 2.666666667 |
| 4 | 2.708333333 |
| 5 | 2.716666667 |
| 6 | 2.718055556 |
| 7 | 2.718253968 |
| 8 | 2.718278770 |
| 9 | 2.718281526 |
| 10 | 2.718281801 |
| 11 | 2.718281826 |
| 12 | 2.718281828 |

## Example: A Second Order Differential Equation

$$\mathbf{y'' + xy' + y = 0}$$

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \ldots$$

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$$y' = a_1 + 2a_2 x + ... + na_n x^{n-1} + (n+1)a_{n+1} x^n + (n+2)a_{n+2} x^{n+1}$$

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$$xy' = a_1x + 2a_2x^2 + .. + na_nx^n + (n+1)a_{n+1}x^{n+1} + (n+2)a_{n+2}x^{n+2} + ...$$

$$y'' = 2a_2 + ... + n(n-1)a_nx^{n-2} + n(n+1)a_{n+1}x^{n-1} + (n+1)(n+2)a_{n+2}x^n + ..$$

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$$y' = a_1 + 2a_2 x + \ldots + n a_n x^{n-1} + (n+1) a_{n+1} x^n + (n+2) a_{n+2} x^{n+1}$$

$$xy' = a_1 x + 2a_2 x^2 + .. + n a_n x^n + (n+1) a_{n+1} x^{n+1} + (n+2) a_{n+2} x^{n+2} + \ldots$$

$$y'' = 2a_2 + \ldots + n(n-1) a_n x^{n-2} + n(n+1) a_{n+1} x^{n-1} + (n+1)(n+2) a_{n+2} x^n + ..$$

Coefficient of $x^n$ in $y + xy' + y$ is

$$a_n + n a_n + (n+1)(n+2) a_{n+2}$$

$$= a_n(1+n) + (n+1)(n+2) a_{n+2} = (n+1)\left[a_n + (n+2) a_{n+2}\right]$$

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But all coefficients must be 0. Thus

Our Equation: $y'' + xy' + y = 0$ has solution

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Thus

$$a_2 = -\frac{a_0}{2}, a_4 = -\frac{a_2}{4} = +\frac{a_0}{2 \cdot 4}, a_6 = -\frac{a_0}{2 \cdot 4 \cdot 6}, ...$$

$$a_3 = \frac{a_1}{3}, a_5 = +\frac{a_1}{3 \cdot 5}, a_7 = -\frac{a_1}{3 \cdot 5 \cdot 7}, ...$$

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$$a_3 = \frac{a_1}{3}, a_5 = +\frac{a_1}{3 \cdot 5}, a_7 = -\frac{a_1}{3 \cdot 5 \cdot 7}, \dots$$

We can write solution as

$$y = a_0 \left[ 1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 - \frac{1}{2 \cdot 4 \cdot 6} x^6 + \dots + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot \dots \cdot 2n} x^{2n} + \dots \right]$$
$$+ a_1 \left[ x - \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 + \dots + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot \dots \cdot (2n-1)} x^{2n-1} + \dots. \right]$$

where $a_0 = y(0)$ and $a_1 = y'(0)$.

Our Equation: $y'' + xy' + y = 0$ has solution

$$y = a_0 \left[ 1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + ... + \frac{(-1)^n}{2 \cdot 4 \cdot 6 \cdot ... \cdot 2n}x^{2n} + ... \right]$$

$$+ a_1 \left[ x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + ... + \frac{(-1)^{n+1}}{3 \cdot 5 \cdot 7 \cdot ... \cdot (2n-1)}x^{2n-1} + .... \right]$$

$$y = a_0 f(x) + a_1 g(x)$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2 \cdot 4 \cdot 6 \cdot ... \cdot 2n}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{3 \cdot 5 \cdot 7 \cdot ... \cdot (2n-1)}$$

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Note first: $a_0 = \frac{2}{5}$

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Coefficient of $x^n$ in $-5y$ is $-5a_n$

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So $a_1 = 13 + 5a_0 = 13 + 5\left(\frac{2}{5}\right) = 15 = 3 \times 5$

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Thus Recurrence Relation is

$$a_{n+1} = \frac{5a_n}{n+1}, \text{ for } n \geq 1$$

| $n$ | $a_{n+1} = \frac{5a_n}{n+1}$ |
|---|---|
| 1 | $a_2 = \frac{1}{2}(5a_1) = \frac{1}{2}(5 \times 3 \times 5) = 3 \times \frac{5^2}{2!}$ |
| 2 | $a_3 = \frac{1}{3}(5a_2) = \frac{1}{3}(5 \times 3 \times \frac{5^2}{2!}) = 3 \times \frac{5^3}{3!}$ |
| 3 | $a_4 = \frac{1}{4}(5a_3) = 3 \times \frac{5^4}{4!}$ |
| 4 | $a_5 = \frac{1}{5}(5a_4) = 3 \times \frac{5^5}{5!}$ |
| ... | |
| $n-1$ | $a_n = \frac{1}{n}(5a_{n-1}) = 3 \times \frac{5^n}{n!}$ |

| $n$ | $a_{n+1} = \frac{5a_n}{n+1}$ |
|---|---|
| 1 | $a_2 = \frac{1}{2}(5a_1) = \frac{1}{2}(5 \times 3 \times 5) = 3 \times \frac{5^2}{2!}$ |
| 2 | $a_3 = \frac{1}{3}(5a_2) = \frac{1}{3}(5 \times 3 \times \frac{5^2}{2!}) = 3 \times \frac{5^3}{3!}$ |
| 3 | $a_4 = \frac{1}{4}(5a_3) = 3 \times \frac{5^4}{4!}$ |
| 4 | $a_5 = \frac{1}{5}(5a_4) = 3 \times \frac{5^5}{5!}$ |
| ... | |
| $n-1$ | $a_n = \frac{1}{n}(5a_{n-1}) = 3 \times \frac{5^n}{n!}$ |

Thus $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + ...a_nx^n + ..$

$$= \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

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$$= \frac{2}{5} + 3 \times \left( 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ... \right)$$

$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

$$= \frac{2}{5} + 3 \times \left(5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$= \frac{2}{5} + 3 - 3 + 3 \times \left(5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

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$$= \frac{2}{5} + 3 - 3 + 3 \times \left(5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$= \frac{2}{5} - 3 + 3 \times \left(1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + ... + \frac{(5x)^n}{n!} + ...\right)$$

$$y = \frac{2}{5} + 3 \times 5x + 3 \times \frac{(5x)^2}{2!} + 3 \times \frac{(5x)^3}{3!} + ... + 3 \times \frac{(5x)^n}{n!} + ...$$

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$$\text{so } y = -\frac{13}{5} + 3e^{5x}$$

**Example: Solve $y' = 5y + 13$ with $y(0) = \frac{2}{5}$ as First Order Linear Differential Equation.**

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Using $y(0) = \dfrac{2}{5}$ yields $C = 3$

## Frequently Encountered Second Order Differential Equations in Applications

$$\mathbf{P(x)y'' + Q(x)y' + R(x)y = 0}$$

| | |
|---|---|
| Airy | $y'' - xy = 0$ |
| Bessel | $x^2y'' + xy' + (x^2 - \nu^2)y = 0$ |
| Chebyshev | $(1 - x^2)y'' - xy' + \alpha^2 y = 0$ |
| Hermite | $y'' - 2xy' + \lambda y = 0$ |
| Laguerre | $xy'' + (1 - x)y' + \lambda y = 0$ |
| Legendre | $(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0$ |

| Equation | Areas of Application |
|---|---|
| Airy | Acoustics, Fiber Optics |
| Bessel | Acoustics, Electrodynamics |
| Chebyshev | Approximation Theory |
| Hermite | Quantum Mechanics |
| Laguerre | Approximation Theory |
| Legendre | Heat Flow, Electrodynamics |

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