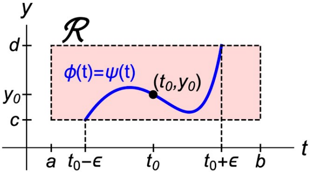
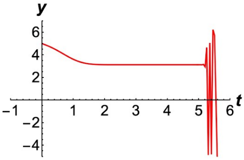
MATH 226 Differential Equations

Intro to Existence and Uniqueness Theorems for ODEs





Class 35: Friday May 9, 2025

Handouts

Self Evaluation

Peer Evaluation

MATH 226 Sticker

Announcements

Project 3 Due Today

Course Response Forms
In Class Next Monday
Bring Laptop/SmartPhone

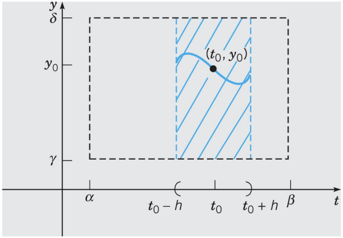
Final Exam
Friday, May 16: 9 - Noon
A - J: Warner 010
K - Z: Warner 100

Basic Existence-Uniqueness Theorem For Differential Equations

Theorem 2.4.2 in Brannan and Boyce

Let the functions f and $\partial f/\partial y$ be continuous in some rectangle $\alpha < t < \beta, \gamma < y < \delta$ containing the point (t_0, y_0). Then, in some interval $t_0 - h < t_0 < t_0 + h$ contained in (α, β) there is a unique solution of the initial value problem

$$y'(t) = f(t, y), \;\; y(t_0) = y_0$$



Example 1: Need For Continuity of $f(t, y)$

$$y'(t) = \frac{1}{t}, y(0) = 0$$

If we try to solve:

$$\int y'(t)\, dt = \int \frac{1}{t}\, dt$$

$$y(t) = \ln|t| + C$$

Theorem: If $f(t, y)$ is continuous in a neighborhood of (a, b), there **exists** a solution of $y' = f(t, y),\ y(a) = b$.

Example 2: Continuity of Partial with Respect to y of $f(t, y)$

$$y' = y^{1/3},\ y(0) = 0$$

One Solution: $y(t) = 0$ for all t

Solve By Separation of Variables: $y^{-1/3}\, y = 1$ yields

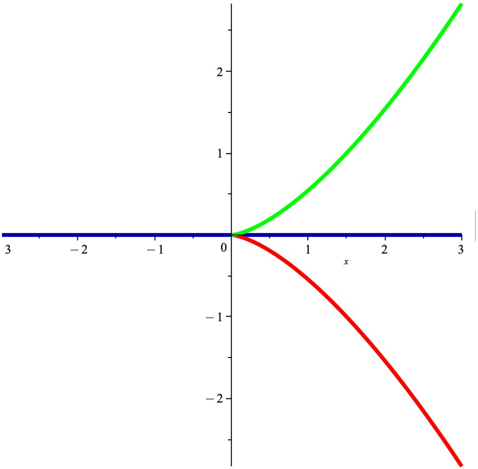
$$\frac{3}{2} y^{2/3} = t + C$$

Initial Condition gives $C = 0$

$$y = \pm \left(\frac{2t}{3} \right)^{3/2}$$

Three Solutions of $y' = y^{1/3},\ y(0) = 0$

$$y = \{0, \left(\frac{2t}{3}\right)^{3/2}, -\left(\frac{2t}{3}\right)^{3/2}\}$$



Solving $y'(t) = f(t, y)$ with $y(0) = 0$ yields

$$y(t) = \int_0^t f(s, y)\, ds$$

We use **Picard's Iteration Method** to get at the integral.

$$\phi_0(t) = 0, \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)))\, ds$$

Charles Émile Picard

July 24, 1856 – December 11, 1941

Biography

Solving $y'(t) = f(t, y)$ with $y(0) = 0$ yields

$$y(t) = \int_0^t f(s, y)\, ds$$

Picard's Iteration Method

$$\phi_0(t) = 0, \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s)))\, ds$$

$$\phi_0(t) = 0, \phi_{n+1}(t) = \int_0^t f(s, \phi_n(s))\, ds$$

- Step 1: $\phi_n(t)$ exists for all n and is continuous.
- Step 2: $\phi(t) = \lim_{n\to\infty} \phi_n(t)$ exists for all t and is continuous.
- Step 3: $\phi(t)$ satisfies the diiferential equation and initial condition.
- Step 4: ϕ is the unique solution.

Example of Picard Iteration

$$y' = 2(y+1),\ y(0) = 0$$

$$\phi_0(t) = 0$$

$$\phi_1(t) = \int_{s=0}^{s=t} f(s, \phi_0(s))\,ds = \int_{s=0}^{s=t} 2(\phi_0(s)+1)\,ds = \int_{s=0}^{s=t} 2\,ds = 2t$$

$$\phi_2(t) = \int_0^t f(s, \phi_1(s))\,ds = \int_0^t 2(\phi_1(s)+1)\,ds = \int_0^t 2(2t+1)\,ds = \int_0^t (4t+2)\,ds = 2t^2 + 2t$$

$$\phi_3(t) = \int_0^t f(s, \phi_2(s))\,ds = \int_0^t 2(\phi_2(s)+1)\,ds = \int_0^t 2(2t^2+2t+1)\,ds = \int_0^t (4t^2+4t+2)\,ds = \frac{4}{3}t^3 + 2t^2 + 2t$$

$$\phi_4(t) = \int_0^t f(s, \phi_3(s))\,ds = \int_0^t 2(\phi_3(s)+1)\,ds = \int_0^t 2(\frac{4}{3}t^3 + 2t^2 + 2t + 1)\,ds = \int_0^t (\frac{8}{3}t^3 + 4t^2 + 4t + 2)\,ds = \frac{2}{3}t^4 + \frac{4}{3}t^3 + 2t^2 + 2t$$