**Math 302 - Abstract Algebra**
Sample Exam 1

This exam is closed-book and closed-notebook. Justify all of your work on these problems. Please write your answers on separate, unused paper and write and sign the honor pledge at the top of your first sheet of paper when you are finished.

1. Suppose that $\sim$ is an equivalence relation on a set $X$. Let $[x]$ denote the equivalence class containing $x$. In other words,
$$[x] = \{y \in X \mid x \sim y\}.$$
Suppose that $a, b, c \in X$ are such that $a \in [b]$ and $a \in [c]$. Show that $[b] \subseteq [c]$.

2. If $a$ and $b$ are integers and $n$ is a positive integer, prove that if $n$ divides $a - b$, then $a \bmod n = b \bmod n$.

3. Let $H = \{A \in GL(2, \mathbb{R}) \mid \det A \text{ is an integer}\}$. Is $H$ a subgroup of $GL(2, \mathbb{R})$? If so, prove that it is, if not, explain why not.

4. (a) Define what it means for an element $a \in G$ to have infinite order.
   (b) Let $G$ be a group and suppose that $a \in G$ has *finite* order. Prove that if $x$ is any other element in $G$, then $|xax^{-1}|$ divides $|a|$. (Note: in fact, using a symmetric argument, it is possible to show that $|a|$ divides $|xax^{-1}|$ so actually, $|xax^{-1}| = |a|$.)

5. Let $G$ be a group and suppose that $\varphi : G \to G$ is a function satisfying $\varphi(gh) = \varphi(g)\varphi(h)$ for all $g, h \in G$. Use induction to prove that for any $g \in G$, $\varphi(g^n) = (\varphi(g))^n$ for all $n \geq 2$.

6. Suppose that $G = \langle a \rangle$ and $|G| = 12$.
   (a) How many subgroups does $G$ have?
   (b) How many elements of order 4 does $G$ have?