Equivalence Relations

↳ common throughout math. Gives us a way to **partition** a set.

Defn A relation $\sim$ on a set X is an **equivalence relation** if

1. $a \sim a$ $\forall a \in X$ (reflexive)
2. if $a \sim b$, then $b \sim a$ (symmetric)
3. if $a \sim b$ and $b \sim c$, then $a \sim c$ (transitivity)

Ex Let $X = \mathbb{Z}$. Say $a \sim b$ if $3 \mid a - b$.

Check:

1. Does $3 \mid a - a$ for all a?

   i.e. does $3 \mid 0$? yes: $0 = 3 \cdot 0$ ✓

2. Sps $a \sim b$, so $3 \mid a - b$. Then there exists $k \in \mathbb{Z}$ s.t. $a - b = 3k$. But then $b - a = 3(-k)$. Since $k \in \mathbb{Z}$, $-k \in \mathbb{Z}$ as well, so by defn of divisibility, $3 \mid b - a$, so $b \sim a$. ✓

3. Sps $a \sim b$ and $b \sim c$. So $3 \mid a-b$ and $3 \mid b-c$.

Thus, $\exists$ integers k, ℓ s.t. $a-b = 3k$ and $b-c = 3\ell$.

But then $a-c = a-b+b-c = 3k+3\ell = 3(k+\ell)$.

Since $k, \ell \in \mathbb{Z}$, so is $k+\ell$. Thus $3 \mid a-c$, so we conclude $a \sim c$, as desired.

So yes: $\sim$ is an equivalence rel.

__Ex.__ Deck of cards.

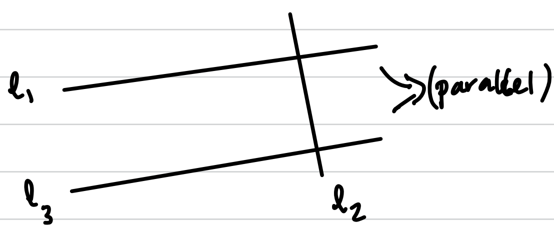
↳ e.g. by suit ♡, ♧, ♤, ◇

↳ __or__ by face value.

__Ex__ For lines ℓ_1, ℓ_2 in $\mathbb{R}^2$, say $\ell_1 \sim \ell_2$ if ℓ_2 intersects ℓ_1.

↳ not an equiv. relation.

↳ not transitive.



if ℓ_1, ℓ_3 parallel, $\ell_1 \not\sim \ell_3$.

<u>Defn</u> For $a \in X$, the <u>equivalence class</u> of a is the set

$$[a] = \{ b \in X \mid a \sim b \}.$$

<u>Ex.</u> cards $[A\heartsuit] = \{\text{all hearts}\}$

Note: $[A\heartsuit] = [K\heartsuit] = [7\heartsuit] = [3\heartsuit] = \ldots$

equal as sets

$A\heartsuit$ and $K\heartsuit$ are <u>representatives</u> of the same equivalence class.