Corollary If $\varphi: G_1 \to G_2$ a homomorphism, then $|\varphi(G_1)|$ divides $|G_1|$ and $|G_2|$.

Why? $\varphi(G_1) \leq G_2$ so $|\varphi(G_1)|$ divides $|G_2|$. (Lagrange's Thm)

Also, $G_1/\ker\varphi \cong \varphi(G_1)$ so $|\varphi(G_1)| = \dfrac{|G_1|}{|\ker\varphi|}$

ie. $|\varphi(G_1)|$ divides $|G_1|$.

Thm Any normal subgroup N of G is the kernel of some homomorphism.

proof: Sps $N \trianglelefteq G$. Consider the homomorphism

$$\varphi: G \to G/N \quad \text{given by} \quad \varphi(g) = gN.$$

φ is a homomorphism because $\varphi(gh) = ghN = gNhN = \varphi(g)\varphi(h)$.

$\ker\varphi$? Identity in G/N is $eN = N$. So $\varphi(g) = gN = N \Leftrightarrow g \in N$. Thus $\ker\varphi = N$.

ie we have a homomorphism for which N is the kernel.

The big idea:

- Factor groups G/H are like cross-sections of G. Help you understand the structure of G.

  - First Isomorphism Theorem says the question of factor groups is equivalent to the question of homomorphic images.

    - In this equivalence, normal subgroups (factor groups) correspond to kernels (homomorphic images).