Properties of homomorphisms

$\varphi(ab) = \varphi(a)\varphi(b)$ $\forall a, b \in G_1$

Sps $\varphi : G_1 \to G_2$ is a homomorphism.

1. $\varphi(e_1) = e_2$

$$\varphi(e_1) = \varphi(e_1 e_1) = \varphi(e_1)\varphi(e_1)$$ cancel $\varphi(e_1)$ both sides

$$\Rightarrow \quad e_2 = \varphi(e_1) \checkmark$$

2. $\varphi(g^n) = [\varphi(g)]^n$ for all $n \in \mathbb{Z}$.

$\hookrightarrow$ $\varphi(\underbrace{g \cdots g}_{n \text{ times}}) = \underbrace{\varphi(g) \cdots \varphi(g)}_{n \text{ times}}$ $\rightsquigarrow$ extends to neg powers as well.

3. If $|g| = n$, then $|\varphi(g)|$ divides n. (Not necessarily equal)

$$g^n = e_1 \Rightarrow \varphi(g^n) = \varphi(e_1) \Rightarrow [\varphi(g)]^n = e_2.$$

$\hookrightarrow$ result follows from corollary of result about cyclic groups.

$\{k \in G_1 \mid \varphi(k) = e_2\}$

4. $\ker \varphi \leq G_1$

One-step subgroup test:

$\ker\varphi$ nonempty because $e_1 \in \ker \varphi$.

Sps $a, b \in \ker\varphi$. Then

$a, b \in \ker\varphi$

$$\varphi(ab^{-1}) = \varphi(a)\varphi(b^{-1}) = \varphi(a)[\varphi(b)]^{-1} = e_2 e_2^{-1}$$
$$= e_2$$

So $ab^{-1} \in \ker\varphi$ and thus $\ker\varphi \leq G_1$.