Properties of homomorphisms and subgroups

Sps $\varphi : G_1 \to G_2$ is a homomorphism and $H \leq G_1$.

"image of H under φ"

1. $\varphi(H) = \{ \varphi(h) \mid h \in H \} \leq G_2$

(one-step subgroup test)

If H nonempty, then $\varphi(H)$ is nonempty.

Sps $\varphi(h_1), \varphi(h_2) \in \varphi(H)$.

$h_1, h_2 \in H$ generic elements of $\varphi(H)$.

Then

$$\varphi(h_1)[\varphi(h_2)]^{-1} = \varphi(h_1)\varphi(h_2^{-1})$$
$$= \varphi(h_1 h_2^{-1})$$
$$\in \varphi(H) \text{ because } h_1 h_2^{-1} \in H. \checkmark$$

but not necessarily same order

2. If H is cyclic, then $\varphi(H)$ is cyclic.

Sps $H = \langle h \rangle$. Then

$$\varphi(H) = \{ \varphi(h^n) \mid n \in \mathbb{Z} \}$$

$$= \{ [\varphi(h)]^n \mid n \in \mathbb{Z} \}$$

$$= \langle \underbrace{\varphi(h)}_{\text{generator.}} \rangle$$

exercise.

3. If H is abelian, then $\varphi(H)$ is abelian.

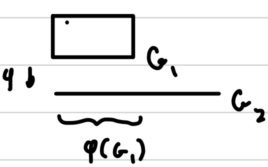
normal

not necessarily normal in G_2. *

4. If $H \trianglelefteq G_1$, then $\varphi(H) \trianglelefteq \varphi(G_1)$ *

{ $aH = Ha$ $\forall a \in$ group

$H \trianglelefteq G_1 \Rightarrow xH = Hx$ for all $x \in G_1$



$\Rightarrow \varphi(xH) = \varphi(Hx)$ for all $x \in G_1$

$\Rightarrow \varphi(x)\varphi(H) = \varphi(H)\varphi(x)$ $\forall x \in G_1$, i.e. all $\varphi(x) \in \varphi(G_1)$

$\Rightarrow \varphi(H) \trianglelefteq \varphi(G_1)$.

5. If $|H| = n$, then $|\varphi(H)|$ divides n.

$\varphi : G_1 \to G_2$

$\varphi|_H : H \to \varphi(H)$, a homomorphism.

"φ restricted to H"

Sps $|\ker \varphi|_H| = m$. Then $\varphi|_H$ is an m-to-1 mapping.

i.e. $|\varphi(H)| \cdot m = |H|$, so $|\varphi(H)|$ divides $|H|$.