The First Isomorphism Theorem

Thm (First Isomorphism Theorem)

Sps $\varphi: G_1 \to G_2$ is a homomorphism.

Let $\varphi(G_1) = \{\varphi(g) \mid g \in G_1\} \subseteq G_2$ = image of G_1 in G_2 under φ.

Then

$$G_1/\ker\varphi \cong \varphi(G_1).$$

The isomorphism is given by $\Phi(g\ker\varphi) = \varphi(g)$

$$\begin{tikzcd}
 & G_1 \arrow[dl] \arrow[dr, "\varphi"] & \\
G_1/\ker\varphi \arrow[rr, "\Phi"'] & & \varphi(G_1) \le G_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & g \arrow[dl, bend right, mapsto] \arrow[dr, bend left, mapsto] & \\
g\ker\varphi \arrow[rr, bend right, mapsto] & & \varphi(g)
\end{tikzcd}$$

$\varphi(G_1) = G_2$

Note: if $\varphi: G_1 \to G_2$ is onto, then $G_1/\ker\varphi \cong G_2$.

G_1 G_2 recall previous example/proof... same as proof of 1st isom thm.

Ex $\mathbb{Z}/8\mathbb{Z} \approx \mathbb{Z}_8$

ker φ

Define $\varphi: \mathbb{Z} \to \mathbb{Z}_8$ by $\varphi(a) = a \bmod 8$

Note: domain is $\mathbb{Z}$, not $\mathbb{Z}/8\mathbb{Z}$

Then φ is a homomorphism:

$$\varphi(a+b) = (a+b) \bmod 8$$

this holds by earlier result

$$= [a \bmod 8 + b \bmod 8] \bmod 8 = \varphi(a) + \varphi(b)$$

group operation in $\mathbb{Z}_8$.

Further, $\ker \varphi = 8\mathbb{Z}$

Finally, φ is onto

Thus, by first isomorphism theorem, $\mathbb{Z}/8\mathbb{Z} \approx \mathbb{Z}_8$

$G_1/\ker\varphi$ $\varphi(G_1)$

BIG IDEA: Can prove the theorem once, in full generality, then apply it in many different situations.