

# The First Isomorphism Theorem

Thm (First Isomorphism Theorem)

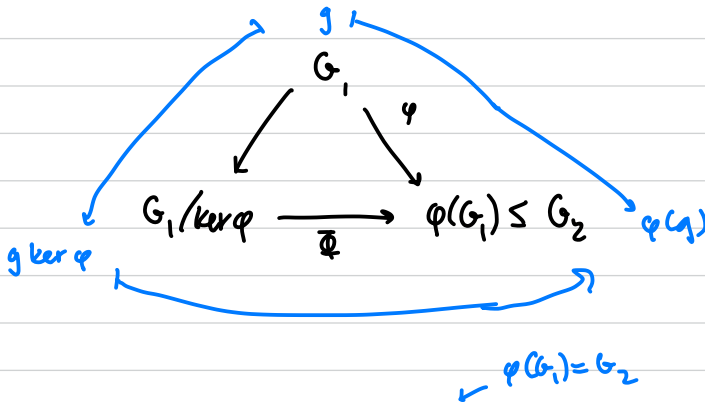
Sp.  $\varphi: G_1 \rightarrow G_2$  is a homomorphism.

Let  $\varphi(G_1) = \{\varphi(g) \mid g \in G_1\} \stackrel{\subset G_2}{=} \text{image of } G_1 \text{ in } G_2 \text{ under } \varphi.$

Then

$$G_1 / \ker \varphi \cong \varphi(G_1).$$

The isomorphism is given by  $\bar{\varphi}(g \ker \varphi) = \varphi(g)$



Note: If  $\varphi: G_1 \rightarrow G_2$  is onto, then  $G_1 / \ker \varphi \cong G_2$

$G_1$   $\swarrow$   
Ex  $\mathbb{Z}/8\mathbb{Z} \approx \mathbb{Z}_8$   
 $\uparrow \ker \varphi$

$G_2$   $\downarrow$   
 recall previous example/proof...  
 same as proof of 1st Ism thm.

Define  $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}_8$  by  $\varphi(a) = a \bmod 8$

$\leftarrow$  Note: domain is  $\mathbb{Z}$ , not  $\mathbb{Z}/8\mathbb{Z}$

Then  $\varphi$  is a homomorphism:

$$\varphi(a+b) = (a+b) \bmod 8$$

this holds by  
earlier result

$$= [a \bmod 8 + b \bmod 8] \bmod 8 = \varphi(a) + \varphi(b)$$

group operation in  $\mathbb{Z}_8$

Further,  $\ker \varphi = 8\mathbb{Z}$

Finally,  $\varphi$  is onto

Thus, by first isomorphism theorem,  $\mathbb{Z}/8\mathbb{Z} \approx \mathbb{Z}_8$

$\uparrow \uparrow$   
 $G/\ker \varphi$   $\uparrow \varphi(G_1)$

BIG IDEA: Can prove the theorem once, in full generality,  
then apply it in many different situations.