<u>Defn</u> Let R be a commutative ring with unity. If every nonzero $a \in R$ has a multiplicative inverse, R is a <u>field</u>.

Notes:

1. $U(F)$ (units in F, i.e. all nonzero elements of F) is a group under multiplication in F.

   thus all group facts hold for $U(F)$.

2. Every field is an integral domain:

   Sps $a \neq 0$. (NTS: a not a zero divisor)

   a is an elt of a field.

   $ab = 0 \Rightarrow a^{-1}ab = 0 \Rightarrow b = 0.$

   NTS: $b = 0$.

<u>Ex</u> $\mathbb{Z}_7$ is a field.

$$\begin{array}{cccccccc}
\mathbb{Z}_7 = \{0, & 1, & 2, & 3, & 4, & 5, & 6\} & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 & 1 & 4 & 5 & 2 & 3 & 6 & \leftarrow \text{corresponding inverses}
\end{array}$$

<u>Ex</u> $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are fields.

<u>Ex</u> $\mathbb{Z}_2[i] = \{a+bi \mid a, b \in \mathbb{Z}_2\}$

$$= \{0+0i,\ 1+0i,\ 0+1i,\ 1+1i\}$$

is not a field: $1+i$ has no inverse

$$1(1+i) = 1+i$$

$$i(1+i) = i + i^2 = i + \overset{\text{additive inverse of } 1 \in \mathbb{Z}_2}{(-1)} = i+1$$

$$(1+i)(1+i) = 1 + \overset{0 \text{ in } \mathbb{Z}_2}{(1+1)}i + i^2 = 1+0i+1 = 0.$$