Ideals

Defn An ideal in a ring R is a subring such that $ra \in I$ and $ar \in I$ for all $a \in I$ and $r \in R$.

↳ Ideal like a vacuum... pulls things in

Note: Ideals play a role in ring theory similar to normal subgroups in group theory.

Ex $2\mathbb{Z} \subset \mathbb{Z}$ is an ideal.

↳ $\underbrace{(2n)}_{\in 2\mathbb{Z}}\underbrace{(m)}_{\in \mathbb{Z}} = \underbrace{2(nm)}_{\in 2\mathbb{Z}}$ and similarly $(m)(2n) \in 2\mathbb{Z}$.

Ex $n\mathbb{Z} \subset \mathbb{Z}$ is an ideal.

↳ Same reasoning.

Ex Sps R is a nontrivial ring.

Consider $S = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \middle| \, a \in R \right\} \subset M_2(R)$

Observe:

S is a subring of $M_2(R)$.

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}$$

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} - \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a-b & 0 \\ 0 & a-b \end{bmatrix}$$

S is not an ideal in $M_2(R)$.

Observe: sps $a \in R$, $a \neq 0$, $a^2 \neq 0$

$$\underbrace{\begin{bmatrix} a & a \\ a & a \end{bmatrix}}_{\in M_2(R)} \underbrace{\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}}_{\in S} = \underbrace{\begin{bmatrix} a^2 & a^2 \\ a^2 & a^2 \end{bmatrix}}_{\notin S.} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$