Thm (Ideal Test) A nonempty subset $I \subseteq R$ is an ideal if

- $a - b \in I$ for all $a, b \in I$ and
- $ra \in I$ and $ar \in I$ for all $a \in I$ and $r \in R$

Proof: This is the subring test, with the additional requirement that $ar \in I$ and $ra \in I$ for all $a \in I$ and all $r \in R$.

Ex Let $a \in R$, where R is commutative. The (principal) ideal generated by a is the set

$$\langle a \rangle = \{ ra \mid r \in R \}$$

(WARNING: careful with notation. Not same as additive subgroup generated by a)

To see that $\langle a \rangle$ is an ideal:

Sps $r_1 a, r_2 a \in \langle a \rangle$.

generic elts of $\langle a \rangle$.

Then $r_1 a - r_2 a = \underbrace{(r_1 - r_2)}_{\in R} a \in \langle a \rangle$. ✓

Now, sps. $r_1 a \in \langle a \rangle$ and $r \in R$.

generic in $\langle a \rangle$ generic in R

Then $r(r_1 a) = (\underbrace{r r_1}_{\in R}) a \in \langle a \rangle$. ✓

Also, since R is commutative,

$(r_1 a) r = r(r_1 a) = (\underbrace{r r_1}_{\in R}) a \in \langle a \rangle$ ✓