Factor Rings

will need proof

Thm Sps R is a ring and I is an ideal. The set of all (additive) cosets of I in R

$$R/I = \{ r + I \mid r \in R \}$$

is a ring under addition defined by

$$(r+I) + (s+I) = (r+s) + I$$

like $(aH)(bH) = abH$ for groups in additive notation

and multiplication defined by

$$(r+I)(s+I) = rs + I.$$

first, restrict attention to group properties

proof: Since R is an abelian group, I is a normal subgroup. So from previous work, R/I is an abelian group under coset addition.

(NTS: coset multiplication is well-defined.)

Sps. $r_1 + I = r_2 + I$ and $s_1 + I = s_2 + I$.

↳ Need: $r_1 s_1 + I = r_2 s_2 + I$.

Since $r_1 + I = r_2 + I$, $r_1 = r_2 + a_r$ for some $a_r \in I$.

Since $s_1 + I = s_2 + I$, $s_1 = s_2 + a_s$ for some $a_s \in I$.

So:

$$r_1 s_1 = (r_2 + a_r)(s_2 + a_s)$$

$$= r_2 s_2 + \underbrace{a_r s_2 + r_2 a_s + a_r a_s}_{\in I}$$

$$= r_2 s_2 + b \quad \text{where } b \in I.$$

Thus $\underbrace{r_1 s_1 - r_2 s_2}_{b} \in I$ so $\xrightarrow{\text{litmus test}}$ $r_1 s_1 + I = r_2 s_2 + I$.

↳ So: coset mult well-defined.

Finally, multiplicative associativity and distribution hold in R/I because they hold in R.