Note: R/I is a ring (with operations defined above) if and only if I is an ideal in R.

Why? Suppose $I \subset R$ is a subring but not an ideal. Then there exists $r \in R$ and $a \in I$ such that $ar \notin I$ or $ra \notin I$.

focus on this possibility

But then $0 + I = a + I$, but multiplication is not well-defined:

$$(0+I)(r+I) = 0+I \neq ar + I = (a+I)(r+I)$$

aka I.

$\neq I$ because $ar \notin I$.

Ex In $\underset{\text{reals}}{\mathbb{R}}[x] = \{ \overbrace{a_0 + a_1 x + \dots + a_n x^n}^{f(x)} \mid a_i \in \mathbb{R} \}$

$$\langle x \rangle = \{ f(x)x \mid f(x) \in \mathbb{R}[x] \}$$

$$= \{ a_1 x + \dots + a_n x^n \mid a_i \in \mathbb{R} \}$$

for any polynomial $f(x)$, $f(x)x$ $\cancel{x f(x)}$ has no constant term

$$\mathbb{R}[x] / \langle x \rangle = \{ f(x) + \langle x \rangle \mid f(x) \in \mathbb{R}[x] \}$$

choose an equivalent representative for coset.

$$= \{ a_0 + \langle x \rangle \mid a_0 \in \mathbb{R} \}$$

$$\approx \mathbb{R}.$$

equivalent (isomorphic!) as rings.

intuitive

Ex $\mathbb{Z} / \underset{\text{ideal in } \mathbb{Z}}{n\mathbb{Z}} = \{ m + n\mathbb{Z} \mid m \in \mathbb{Z} \}$

$$\approx \mathbb{Z}_n \quad \text{(as rings)}$$

$$(m_1 + n\mathbb{Z})(m_2 + n\mathbb{Z}) = m_1 m_2 + n\mathbb{Z} = (m_1 m_2) \bmod n + n\mathbb{Z}$$

equiv. coset representative.