Ring Homomorphisms

Defn A map φ from a ring R_1 to a ring R_2 is a ring homomorphism if

$$\varphi(a+b) = \varphi(a) + \varphi(b)$$

$$\varphi(ab) = \varphi(a)\varphi(b) \quad \text{for all } a,b \in R_1$$

A ring homomorphism is called a ring isomorphism if it is 1-1 and onto.

The kernel of a ring homomorphism φ is the additive kernel:

$$\ker \varphi = \{ r \in R_1 \mid \varphi(r_1) = 0_{R_2} \}$$

Ex $\varphi: \mathbb{Z} \to \mathbb{Z}_n$ defined by

$$\varphi(x) = x \bmod n$$

is a ring homomorphism:

- we've already shown $\varphi(x+y) = \varphi(x) + \varphi(y)$
- $\varphi(xy) = (xy) \bmod n = \underbrace{[\underbrace{(x \bmod n)}_{\varphi(x)}\underbrace{(y \bmod n)}_{\varphi(y)}] \bmod n}_{\text{mult. in } \mathbb{Z}_n}$

Ex. $\varphi: \mathbb{Z} \to 2\mathbb{Z}$

$$\varphi(x) = 2x$$

- $\varphi(x+y) = 2(x+y) = 2x+2y = \varphi(x) + \varphi(y)$.

But note!

mult not preserves

- $\varphi((1)(3)) = \varphi(3) = 6 \neq 12 = (2)(6) = \varphi(1)\varphi(3)$

Thus: φ is <u>not</u> a ring homomorphism.