__Ex__ $\varphi : \mathbb{R}[x] \to \mathbb{R}$ defined by

$$\varphi(f(x)) = f(0)$$

generic polynomial in $\mathbb{R}[x]$

substitute 0 for x

Note: if $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$

then $f(0) = a_0$ ← constant term

$\varphi(f(x))$

$\varphi(g(x))$

Sps $g(x) = b_0 + b_1 x + \dots + b_m x^m$. Similarly $g(0) = b_0$.

Then

$(a_0+b_0) + (a_1+b_1)x$ + higher-order terms.

$$\varphi(f(x)+g(x)) = a_0 + b_0 = \varphi(f(x)) + \varphi(g(x)). \checkmark$$

(preserves addition)

and

$a_0 b_0 + (a_1 b_0 + a_0 b_1)x$ + higher order terms.

$$\varphi(f(x)g(x)) = a_0 b_0 = \varphi(f(x))\varphi(g(x)) \checkmark$$

(preserves multiplication)

constant term of product is $a_0 b_0$

So φ is a ring homomorphism.

Note: The previous example is a special case of a more generally phenomenon:

Sps R is a ring, and $a \in R$.

Define $\varphi: R[x] \to R$ by

polynomial ring with coefficients in R

substitute a for x

$$\varphi(f(x)) = f(a).$$

this will be a sum of products of elements in R, thus in R.

Then φ is a ring homomorphism, called the <u>evaluation map at a</u>.

exercise