Properties of Ring Homomorphisms

Sps $\varphi : R_1 \to R_2$ is a ring homomorphism.

1. $\varphi(\underset{r+\cdots+r,\ n \text{ times}}{n \cdot r}) = n \cdot \varphi(r)$ for all $r \in R_1$, $n \in \mathbb{Z}$.

   $\varphi(r^n) = [\varphi(r)]^n$ for all $r \in R_1$ and any positive integer n.

2. $S \subset R_1$ a subring $\Rightarrow$ $\underset{\text{a set}}{\varphi(S)}$ a subring of R_2.

   ↳ subring test:

   Sps $\varphi(a), \varphi(b) \in \varphi(S)$. Then

   $$\varphi(a) - \varphi(b) = \varphi(\underbrace{a-b}_{\in S}) \in \varphi(S) \quad \text{and} \quad \varphi(a)\varphi(b) = \varphi(\underbrace{ab}_{\in S}) \in \varphi(S)$$

3. I an ideal in R_1 and φ onto $\Rightarrow$ $\varphi(I)$ an ideal in R_2.

Ideal test:

know $\varphi(I)$ a subring of R_2. Sps $x \in \varphi(I)$ and $s \in R_2$.

(NTS: $sx, xs \in \varphi(I)$.)

But $x = \varphi(a)$ for some $a \in I$ and, since φ is onto

$s = \varphi(r)$ for some $r \in R_1$.

Then $sx = \varphi(r)\varphi(a) = \varphi(ra)$.

Since I is an ideal, $ra \in I$, so $sx = \varphi(I)$.

(similar for xs).