4. $J \subset R_2$ an ideal $\Rightarrow \varphi^{-1}(J)$ an ideal in R_1.

$\hookrightarrow \{r \in R_1 \mid \varphi(r) \in J\} \subset R_1$.

$\hookrightarrow$ ideal test:

Sps $r_1, r_2 \in \varphi^{-1}(J)$, $r_3 \in R_1$. (NTS: $r_1 - r_2 \in \varphi^{-1}(J)$, $r_3 r_1$ and $r_1 r_3 \in \varphi^{-1}(J)$)

Since $r_1, r_2 \in \varphi^{-1}(J)$, $\varphi(r_1), \varphi(r_2) \in J$.

This implies $\varphi(r_1) - \varphi(r_2) \in J$

$\Rightarrow \varphi(r_1 - r_2) \in J$

$\Rightarrow r_1 - r_2 \in \varphi^{-1}(J)$. ✓

Now, $r_1 \in \varphi^{-1}(J) \Rightarrow \varphi(r_1) \in J$. And $r_3 \in R_1 \Rightarrow \varphi(r_3) \in R_2$.

This implies $\underbrace{\varphi(r_3)}_{\in R}\underbrace{\varphi(r_1)}_{\in J} \in J$, because J an ideal.

$\Rightarrow \varphi(r_3 r_1) \in J$

$\Rightarrow r_3 r_1 \in \varphi^{-1}(J)$. ✓ [similarly $r_1 r_3 \in \varphi^{-1}(J)$]

ab = ba for all $a, b \in R_1$

5. R_1 commutative $\Rightarrow$ $\varphi(R_1)$ commutative

6. If R_1 has unity 1, $R_2 \neq \{0\}$, and φ is onto,

then $\varphi(1)$ is a (the) unity in R_2.

generic element of R

Let $s \in R_2$. Since φ onto, $s = \varphi(r)$ for some $r \in R_1$.

$$\varphi(1)s = \varphi(1)\varphi(r) = \varphi(1r) = \varphi(r) = s.$$

7. φ a ring ~~homomorphism~~ isomorphism $\Leftrightarrow$ φ onto and $\ker\varphi = \{0_{R_1}\}$

additive kernel

If $|\ker\varphi| = n$, φ is an n-to-1 mapping.

↳ follows for same reasons as group homomorphisms because $\ker\varphi$ is additive kernel of φ and R_1, R_2 are groups under addition.

8. $\varphi : R_1 \to R_2$ a ring ~~homomorphism~~ isomorphism.

$\Leftrightarrow$ $\varphi^{-1} : R_2 \to R_1$ a ring ~~homomorphism~~ isomorphism.

$\hookrightarrow$ exercise $\rightsquigarrow$ show φ^{-1} preserves addition and multiplication.