Thm Sps $\varphi : R_1 \to R_2$ is a ring homomorphism. Then $\ker \varphi$ is an ideal in R_1.

proof: Follows from property 4:

$\{0_{R_2}\}$ is an ideal in R_2 because $0_{R_2} s = 0_{R_2}$ for all $s \in R_2$.

So $\varphi^{-1}(\{0_{R_2}\})$ is an ideal in R_1. But $\varphi^{-1}(\{0_{R_2}\}) = \ker \varphi$.

Note: This tells us that if $\varphi : R_1 \to R_2$ is a ring homomorphism, then

$R_1 / \ker \varphi$ is a ring.

<u>Thm</u> (First Isomorphism Theorem for Rings)

Sps $\varphi : R_1 \to R_2$ is a ring homomorphism.

Then

$$R_1 / \ker\varphi \cong \varphi(R_1)$$

The isomorphism is given by $\Phi(r + \ker\varphi) = \varphi(r)$.

Note: <u>if</u> $\varphi : R_1 \to R_2$ is <u>onto</u>, then $R_1/\ker\varphi \cong R_2$.

$$\begin{tikzcd}
 & R_1 \arrow[dl] \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\
R_1/\ker\varphi \arrow[rr] & {} & \varphi(R_1) \subset R_2
\end{tikzcd}$$

$$\begin{tikzcd}
 & r \arrow[dl, mapsto, bend right] \arrow[dr, mapsto, bend left] & \\
r + \ker\varphi \arrow[rr, mapsto, bend right] & & \varphi(r) \\
 & & \Phi(r + \ker\varphi)
\end{tikzcd}$$