Examples of Groups

1. $\mathbb{Z}$ under addition.

closed? ✓

associative? $a+(b+c) \overset{?}{=} (a+b)+c$ ✓ yes.

abelian

identity? $0+n = n+0 = n \quad \forall n \in \mathbb{Z}$. So identity is 0.

inverses? Sps $n \in \mathbb{Z}$. $n+(-n) = (-n)+n = 0$

Since $-n$ is an integer as well, n has an inverse for all n.

2. $\mathbb{Z}$ under multiplication?

closed ✓

associative ✓

identity? $\longrightarrow$ $1n = n1 = n \quad \forall n$ ✓

✗ inverses? Eg: consider $n=2$. $\left(\frac{1}{2}\right)2 = 1$

not in $\mathbb{Z}$.

Further, there is no integer k s.t. $2k=1$.

$\mathbb{Z}$ not closed under inverses. (mult). Not a group.

rationals

3. $\mathbb{Q} - \{0\} = \mathbb{Q}^*$ under multiplication. is a group.

abelian

nonzero rationals.

inverses: $\frac{a}{b}$ $(a \neq 0)$ $\rightsquigarrow$ $\frac{b}{a}$

4. D_3, or D_n for any $n \geq 3$.

called dihedral groups

↳ symmetries of regular n-sided polygons.

nonabelian

size: D_n has $2n$ elts for any n. (finite group)

"remainders" in division algorithm/ modular arithmetic

5. $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ $(n \in \mathbb{Z}, n \geq 1)$

under addition modulo n.

Ex: $\mathbb{Z}_4 = \{0, 1, 2, 3\}$

finite, abelian

identity: 0

inverses:

0	1	2	3	x
↓	↓	↓	↓	
0	3	2	1	x^{-1}

"units"

$n \in \mathbb{Z}, n \geq 1$

6. $U(n) = \{ m \in \mathbb{Z} \mid 1 \leq m < n \text{ and } \gcd(m,n) = 1 \}$

under mult. modulo n.

(proof omitted.)

<u>Ex.</u> $U(10) = \{1, 3, 7, 9\}$

identity : 1

1	3	7	9	x
1	7	3	9	x^{-1}

proving closure and inverses in general is somewhat challenging. (But possible!)