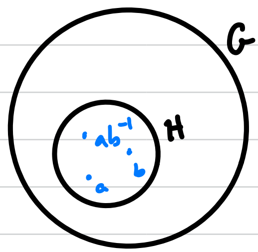
Thm (One-Step Subgroup Test)

Sps H is a nonempty subset of G. If $ab^{-1} \in H$ whenever $a, b \in H$, then H is a subgroup of G.



proof: NTS: identity, inverses, closure.

The identity is in H as follows:

Since $H \neq \overset{\text{empty set}}{\emptyset}$, choose some $x \in H$. Then letting $a = b = x$ by assumption $\underbrace{x}_{a}\underbrace{x^{-1}}_{b^{-1}} \in H$, i.e. $e_G \in H$. ✓

Now, suppose that $x \in H$. By above, $e_G \in H$. Thus, by assumption $\underbrace{e}_{a}\underbrace{x^{-1}}_{b^{-1}} \in H$, i.e. $x^{-1} \in H$, so H is closed under inverses. ✓

Finally, suppose that $x, y \in H$. Then, by above, $y^{-1} \in H$. Thus by assumption $\underbrace{x}_{a}\underbrace{(y^{-1})^{-1}}_{b^{-1}} \in H$, i.e. $xy \in H$, so H is closed under the binary operation. ✓

$\mathbb{Q} - \{0\}$ $\quad$ $\mathbb{R} - \{0\}$

Ex. $\mathbb{Q}^* \leq \mathbb{R}^*$ (under mult.)

$\mathbb{Q}^*$ nonempty? yes! (e.g. $\frac{1}{2} \in \mathbb{Q}^*$)

Sps $\underbrace{\frac{p}{q}}_{a}, \underbrace{\frac{r}{s}}_{b} \in \mathbb{Q}^*$. Then $\left(\frac{r}{s}\right)^{-1} = \frac{s}{r}$

note: $s \neq 0$

Note: $r \neq 0$ b/c $\frac{r}{s} \in \mathbb{Q}^*$

Then $\underbrace{\frac{p}{q} \cdot \frac{s}{r}}_{ab^{-1}} = \frac{ps}{qr}$ neither product is 0 ... so elt of $\mathbb{Q}^*$ ✓

Thus $\mathbb{Q}^* \leq \mathbb{R}^*$.

subgp.