Thm Let H be a nonempty finite subset of a group G that is closed under the binary operation of G. Then H is a subgroup of G.

proof: By Two-Step Subgroup Test, it suffices to show that

$a^{-1} \in H$ whenever $a \in H$. ← strategy.

If $a = e$, since $a^{-1} = a$, we're done.

Sps. $a \neq e$. Consider $a, a^2, a^3, \ldots$ ← positive powers of a.

hypothesis → By closure, all are in H.

pigeon hole principle ↙

But H finite ⇒ $a^m = a^n$ for some $m > n$.

Note $m-n \neq 0$ (b/c $m \neq n$). Also, $m-n \neq 1$ b/c if so

$a^m = a^n \Rightarrow a^{n+1} = a^n \Rightarrow aa^n = a^n \Rightarrow a = e$, a contradiction.

So $m-n \geq 1$. And then,

$a^m = a^n \Rightarrow e = a^{m-n} = a\,a^{m-n-1}$. So $a^{-1} = a^{m-n-1} \in H$. ✓

(a^{m-n-1}: a positive power of a, thus in H.)

(↑ in G, mult both sides by a^{-n}.)