For every group, there are a number of subgroups we can consider.

- We've already seen the trivial subgroup $H = \{e\}$.

- Sps $a \in G$. Let

$$\langle a \rangle = \{a^n \mid n \in \mathbb{Z}\} = \{\ldots a^{-2}, a^{-1}, \overset{e}{a^0}, a, a^2, a^3, \ldots\}$$

$\langle a \rangle$ is called the cyclic subgroup generated by a.

Note: $\langle a \rangle$ can be a finite or an infinite set.

↳ ex $\langle 5 \rangle$ in $\mathbb{Z}_{15}$
$\{0, 5, 10\}$

↳ ex. $\langle 2 \rangle$ in $\mathbb{Z}$
$= \{\text{evens}\}$
$= \{\ldots, -4, -2, 0, 2, 4, 6 \ldots\}$

Thm For each $a \in G$, $\langle a \rangle$ is a subgroup of G.

proof: Use one-step subgroup test:

$\langle a \rangle$ nonempty b/c $a \in \langle a \rangle$

generic elts of $\langle a \rangle$.

Now, sps $a^m, a^n \in \langle a \rangle$. $(m, n \in \mathbb{Z})$.

Then $a^m (a^n)^{-1} = a^m a^{-n} = a^{m-n} \in \langle a \rangle$ b/c $m-n \in \mathbb{Z}$. ✓

- For a group G the <u>center</u> of G is the set

$$Z(G) = \{x \in G \mid xy = yx \text{ for } \underline{\text{all}}\ y \in G\}.$$

↳ set of elements that commute will <u>all</u> other elts in G.

<u>Ex</u> For any G, $e \in Z(G)$ b/c $ey = ye = y$.

<u>Ex.</u> If G abelian, then $Z(G) = G$.

- Sps that G is a group and $a \in G$. The <u>centralizer of a in G</u>, denoted $C(a)$ is

$$C(a) = \{g \in G \mid ga = ag\}$$

↳ set of elements that commute with <u>a</u>.

<u>Ex</u> In D_3, $C(R_{120}) = \{R_0, R_{120}, R_{240}\}$ $\left(= \langle R_{120} \rangle = \langle R_{240} \rangle\right)$

<u>Ex</u> In any group $C(e) = G$.

Thm For any $a \in G$, $C(a) \leq G$

proof: exercise.

Thm For any G, $Z(G) \leq G$.

proof: exercise.