upshot: cyclic groups have tons of easy to see structure

Thm (Fundamental Theorem of Cyclic Groups) FTCG

Consider a cyclic group. Let a be a generator of the group. (Thus the group can be expressed $\langle a \rangle$.)

gives complete understanding of subgp. structure

1. Every subgroup of a cyclic group is cyclic.
2. If $\langle a \rangle$ has order n, the order of every subgroup of $\langle a \rangle$ is a divisor of n.
3. For each divisor d of n, there exists exactly one subgroup of $\langle a \rangle$ of order d, namely $\langle a^{n/d} \rangle$.

Important Observation: This theorem applies to "free-standing" cyclic groups, as well as to any cyclic subgroup of a group G.

The Point: We completely understand the subgroup structure of cyclic (sub)groups.

Note: In general if $|G| = n$ and $d \mid n$, there may or may not be a subgroup $H \leq G$ of order d.

If one does exist, it might not be unique.

so cyclic groups are special in this way.

not cyclic

Ex: $|D_4| = 8$. D_4 has 5 subgroups of order 2.

subgroup order 2 not unique.

Later: we'll see a group with 12 elements and no subgroups of order 6.

Thm (Fundamental Theorem of Cyclic Groups) FTCG

Consider a cyclic group. Let a be a generator of the group. (Thus the group can be expressed $\langle a \rangle$.)

1. Every subgroup of a cyclic group is cyclic.
2. If $\langle a \rangle$ has order n, the order of every subgroup of $\langle a \rangle$ is a divisor of n.
3. For each divisor d of n, there exists exactly one subgroup of $\langle a \rangle$ of order d, namely $\langle a^{n/d} \rangle$.

Ex.

$\langle 1 \rangle$

Subgroups of $\mathbb{Z}_{10}$ are:

FTCG says: $\langle 1 \rangle = \{0, 1, 2, 3, \ldots, 9\}$ $\quad |\langle 1 \rangle| = 10$

$\langle 2 \rangle = \{2, 4, 6, 8, 0\}$ $\quad |\langle 2 \rangle| = 5$

$\langle 5 \rangle = \{5, 0\}$ $\quad |\langle 5 \rangle| = 2$

$\langle 0 \rangle = \{0\}$ $\quad |\langle 0 \rangle| = 1$

FTCG says these are all of the subgroups of $\mathbb{Z}_{10}$. They are all cyclic and there are no hidden others.

Ex. Sps $a \in G$ where $|a| = 10$. Subgroups of $\langle a \rangle$ (which are in turn subgroups of G) are:

$$\langle a^1 \rangle, \langle a^2 \rangle, \langle a^5 \rangle, \langle a^0 \rangle.$$

FTCG says these are all the subgroups of $\langle a \rangle$ (but not necessarily all of the subgroups of G)