Permutation Groups

Intuition:

- cyclic groups are simplest groups.
  - ↳ e.g. based on one element, abelian, etc.
- permutation groups (S_n) are the most general.
  - ↳ e.g. for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$ ... highly nonabelian
  - ↳ in fact, every finite group G of order n can be understood as a subgroup of S_n.

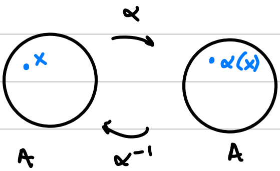
Defn Suppose A is a set. A **permutation** of A is a 1-1, onto function $A \to A$.

Recall: 1. $\alpha, \beta : A \to A$ both 1-1 $\Rightarrow$ $\alpha\beta$ (composite) is 1-1.

2. $\alpha, \beta : A \to A$ both onto $\Rightarrow$ $\alpha\beta$ is onto.

3. α 1-1 and onto $\Rightarrow$ function α^{-1} exists s.t.

$\alpha^{-1}(\alpha(x)) = x$ and $\alpha(\alpha^{-1}(x)) = x$ for all $x \in A$.



So: The collection of permutations of A forms a group under composition.

- * see below · identity? $\varepsilon : A \to A$ (epsilon) given by $\varepsilon(x) = x$ for all $x \in A$. (1-1, onto, so a permutation)
- composition of functions is associative.
- closure from (1) and (2) above.
- each permutation α has an inverse α^{-1} by (3) above.

compare with $R_0 \in D_n$ and $I \in GL(2,\mathbb{R})$

Consider $\varepsilon : A \to A$, $\varepsilon(x) = x$ for all $x \in A$.

↳ in what way is this the identity?

Sps α is a permutation of A.

Then for any $x \in A$

a new function, given by composition

defn of ε

$$(\varepsilon\alpha)(x) = \varepsilon(\underbrace{\alpha(x)}_{\in A}) = \alpha(x).$$

So as permutations (i.e. functions), $\varepsilon\alpha = \alpha$.

Recall: $f, g : X \to X$ are equal as functions if and only if $f(x) = g(x)$ for all $x \in X$.

new function

Similarly, $(\alpha\varepsilon)(x) = \alpha(\varepsilon(x))$

defn of ε

$= \alpha(x)$ for all x

So $\alpha\varepsilon = \alpha$. (as permutations).

↳ So: $\alpha\varepsilon = \alpha = \varepsilon\alpha$

So ε acts as identity in group.

Inverses: (3) above says $\exists\, \alpha^{-1}$ s.t.

$$(\alpha\alpha^{-1})(x) = \alpha(\alpha^{-1}(x)) \underset{(3)}{=} x \quad \forall\, x.$$

i.e. $(\alpha\alpha^{-1})(x) = x = \varepsilon(x)$ for $\forall x$ ⤳ i.e. as permutations, $\alpha\alpha^{-1} = \varepsilon$

similar for $\alpha^{-1}\alpha = \varepsilon$.