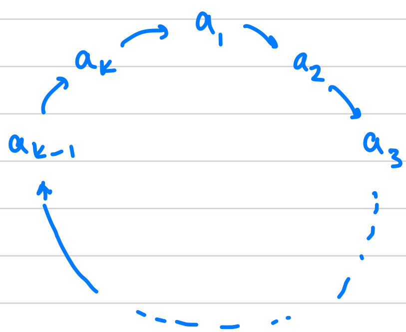
Consider a k-cycle: $(a_1\ a_2\ a_3\ \cdots\ a_k)$

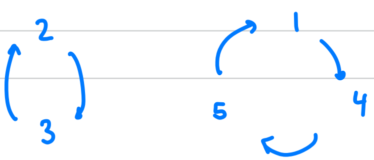
Then $|(a_1\ a_2\ \cdots\ a_k)| = k$.



Thm Sps $\beta, \gamma \in S_n$ (gamma) are disjoint cycles. Then

$$|\beta\gamma| = \operatorname{lcm}(|\beta|, |\gamma|) = \text{lcm of lengths of cycles}$$

Ex. $|(2\,3)(1\,4\,5)| = 6$



... like gears

Before we prove the theorem, recall:

<u>Lemma</u> Sps $s = \text{lcm}(m, n)$. If t is such that $m | t$ and $n | t$, then $s | t$.

(The least common multiple divides any other multiple.)

proof of thm:

Let $t = |\beta\gamma|$ and $s = \text{lcm}(|\beta|, |\gamma|)$

Goal: Show $t | s$ and $s | t$, so $t = s$.

1. To show $t|s$: since β and γ are disjoint, $\beta\gamma = \gamma\beta$ Thus:

$$(\beta\gamma)^s = \beta\gamma\beta\gamma \cdots \beta\gamma = \underbrace{\beta^s \gamma^s}_{} = \varepsilon \cdot \varepsilon = \varepsilon$$

lcm → s is a mult of $|\beta|, |\gamma|$

Since $(\beta\gamma)^s = \varepsilon$, conclude that $|\beta\gamma| \,\big|\, s$, i.e. $t | s$.

$\operatorname{lcm}(|\beta|,|\gamma|)$ $\qquad |\beta\gamma|$

2. To see that $s \mid t$:

Since $|\beta\gamma| = t$, $(\beta\gamma)^t = \varepsilon$.

Since $\beta\gamma = \gamma\beta$, this implies $\beta^t\gamma^t = \varepsilon$.

disjoint cycles

Thus $\beta^t = \gamma^{-t}$.

Since β and γ are disjoint, must be that $\beta^t = \gamma^{-t} = \varepsilon$

(if not, β^t would "move" something not moved by γ^t)

Thus, $\beta^t = \varepsilon$ so $|\beta| \mid t$. Similarly, $|\gamma| \mid t$.

Then, by the lemma $s \mid t$. ✓

For a product of more than 2 disjoint cycles, work inductively:

induction hypothesis

Sps $|\gamma_1\gamma_2 \cdots \gamma_n| = \operatorname{lcm}(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_n|)$.

Then $|\gamma_1\gamma_2 \cdots \gamma_n\gamma_{n+1}| = \operatorname{lcm}(\operatorname{lcm}(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_n|), |\gamma_{n+1}|)$

induction step.

$$= \operatorname{lcm}(|\gamma_1|, |\gamma_2|, \ldots, |\gamma_{n+1}|)$$