FACT: Every permutation in S_n, $n \geq 2$, can be written as a product of 2-cycles (transpositions).

not necessarily disjointly or uniquely

Ex $(1\ 3\ 7)(2\ 4\ 6\ 5) \in S_8$

$= (1\,7)(1\,3)(2\,5)(2\,6)(2\,4)$

$= (1\,3)(3\,7)(2\,4)(4\,6)(6\,5)$

$\varepsilon = (1\,2)(1\,2)$

We use 2-cycle decomposition to classify permutations as even or odd.

↳ this is a well-defined notion because of next theorem.

Thm If $\alpha \in S_n$ can be expressed as a product of an even number of transpositions then every decomposition of α into transpositions will have an even number of transpositions. Similar for odd.

identity

Lemma Every decomposition of ε in 2-cycles has an even number of 2-cycles.

proof of theorem, given the lemma:

Sps $\alpha = \beta_1 \beta_2 \cdots \beta_r = \delta_1 \delta_2 \cdots \delta_s$ where β_i, δ_j are 2-cycles

(NTS: r, s both even or both odd)

Then $\beta_1 \cdots \beta_r \delta_s^{-1} \delta_{s-1}^{-1} \cdots \delta_2^{-1} \delta_1^{-1} = \varepsilon$

$\delta_s^{-1} = \delta_s$ .... two-cycles.

$(a\,b)(a\,b) = \varepsilon$

So, by lemma, $r+s$ is even, thus r, s both even or both odd.